In this paper, Part 7 of the thematic tutorial series “Acoustofluidics – exploiting ultrasonic standing waves, forces and acoustic streaming in microfluidic systems for cell and particle manipulation”, we present the theory of the acoustic radiation force; a second-order, time-averaged effect responsible for the acoustophoretic motion of suspended, micrometre-sized particles in an ultrasound field.

1. Introduction

When an ultrasound field is imposed on a fluid containing a suspension of particles, the latter will be affected by the so-called acoustic radiation force arising from the scattering of the acoustic waves on the particle. The particle motion resulting from the acoustic radiation force is denoted acoustophoresis, and plays a key role in on-chip microparticle handling, as briefly reviewed in Part 1 of the Tutorial Series.¹

The studies of acoustic radiation forces on suspended particles have a long history. The analysis of incompressible particles in acoustic fields dates back to the work in 1934 by King,² while the forces on compressible particles in plane acoustic waves were calculated in 1955 by Yosioka and Kawasima.³ Their work was admirably summarized and generalized in 1962 in a short paper by Gorkov,⁴ and we follow his approach here in deriving the acoustic radiation force, filling in the details originally left out.

The theory of the radiation force relies on a perturbation expansion of the acoustic fields in the fluid. This perturbation theory is treated in detail in Part 2 of the Tutorial Series,⁵ but we summarize the main results. The ultrasound perturbations on a quiescent fluid are considered to first and second order (subscript 1 and 2, respectively) in density ρ, pressure p, and velocity v,

\[
\rho = \rho_0^0 + \rho_1 + \rho_2, \quad (1a)
\]
\[
p = \rho_0^0 c_0^0 v_1 + p_2, \quad (1b)
\]
\[
v = v_1 + v_2, \quad (1c)
\]

where \( c_0 \) is the speed of sound in the fluid, and where \( p_1 = c_0^0 \rho_1 \). Neglecting viscosity in the bulk fluid,⁶ the first-order continuity and Navier–Stokes equations are

\[
\partial_t \rho_1 = -\rho_0 \nabla \cdot v_1, \quad (2a)
\]
\[
\rho_0 \partial_t v_1 = -c_0^2 \nabla p_1. \quad (2b)
\]

We assume time-harmonic fields,

\[
\rho_1(r) = \rho_1(r) e^{-i \omega t}, \quad (3a)
\]
\[
p_1(r) = p_1(r) e^{-i \omega t}, \quad (3b)
\]
\[
v_1(r) = v_1(r) e^{-i \omega t}, \quad (3c)
\]

and introduce the velocity potential, \( \phi_1 \),

\[
v_1(r) = \nabla \phi_1(r), \quad (4a)
\]
\[
p_1(r) = i \rho_0 \omega \phi_1(r), \quad (4b)
\]
\[
\rho_1(r) = \rho_0 e^{i \omega t} \frac{\partial \phi_1(r)}{\partial t}. \quad (4c)
\]

The potential fulfills the wave equation

\[
\nabla^2 \phi_1 = \frac{1}{c_0^2} \frac{\partial^2 \phi_1}{\partial t^2} = -\frac{\omega^2}{c_0^2} \phi_1, \quad (5)
\]

which forms the starting point for the scattering theory used below to calculate the acoustic radiation force acting on the particle.

The observed acoustophoretic motion is not resolved on the \( \mu s \) time scale of the

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**Foreword**

The acoustic radiation force is a key parameter in the field of acoustofluidics and is most commonly utilised in acoustic particle and cell manipulation. In this seventh paper of 23 in the Lab on a Chip tutorial series of Acoustofluidics, Henrik Bruus describes the theory behind the forces acting on single particles in an acoustic field, not taking into account particle–particle interactions that occur at higher particle concentrations. Starting from perturbation theory, the fundamental acoustic radiation force expression is derived and examples for theoretical validation are presented.

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imposed MHz ultrasound wave, but is the result of the radiation force averaged over a full oscillation cycle. Thus, regarding the second-order perturbation terms, we only need to study time averages ⟨X⟩ over a full oscillation period τ of quantities, ⟨X(t)⟩,

\[ \langle X(t) \rangle = \frac{1}{\tau} \int_{0}^{\tau} dt X(t). \]  

(6)

The time-averaged, second-order acoustic pressure ⟨p2⟩ in the inviscid bulk fluid is given by

\[ \nabla \langle p_2 \rangle = -\langle p_1 \partial_t v_1 \rangle - \rho_0 \langle (v_1 \cdot \nabla) v_1 \rangle, \]

(7a)

\[ \langle p_2 \rangle = \frac{1}{2} \kappa_0 \langle \rho_0^2 \rangle - \frac{1}{2} \rho_0 \langle v_1^2 \rangle. \]

(7b)

where in the latter equality we have used eqn (2b) and (4a) to obtain −⟨p_1 ∂_t v_1⟩ = (c_0^2/2ρ_0)∇⟨ρ_0^2⟩ and \( \langle v_1 \cdot \nabla v_1 \rangle = (1/2) \nabla (v_1^2) \), respectively, and introduced the compressibility \( \kappa_0 = 1/(\rho_0 c_0^2) \) of the fluid. We note that the physical, real-valued average \( \langle f(g) \rangle \) of two harmonically varying fields \( f \) and \( g \) with the complex representation eqn (3), is given by the real-part rule

\[ \langle f(g) \rangle = \frac{1}{2} \Re \{ f(r)g^*(r) \}, \]

(8)

where the asterisk denotes complex conjugation.

II. The acoustic radiation force

Below we calculate the acoustic radiation force on a compressible, spherical, micrometre-sized particle of radius \( a \) suspended in an inviscid fluid in an ultrasound field of wavelength \( \lambda \). A small particle, i.e. \( a \ll \lambda \), of density \( \rho_0 \) and compressibility \( \kappa_0 \) acts as a weak point-scatterer of acoustic waves, which can thus be treated by first-order scattering theory. An incoming wave described by some given velocity potential \( \phi_{in} \), results in a scattered wave \( \phi_{sc} \) propagating away from the particle. For sufficiently weak incoming and scattered waves, the total first-order acoustic field \( \phi_1 \) is given by the sum of the two as sketched in Fig. 1(a),

\[ \phi_1 = \phi_{in} + \phi_{sc}, \]

(9a)

\[ v_1 = \nabla \phi_1 = \nabla \phi_{in} + \nabla \phi_{sc}, \]

(9b)

\[ p_1 = i\rho_0 \omega \phi_1 = i\rho_0 \omega \phi_{in} + i\rho_0 \omega \phi_{sc}. \]

(9c)

Once the first-order scattered field \( \phi_{sc} \) has been determined for the given incoming first-order field \( \phi_{in} \), the acoustic radiation force \( F^{rad} \) on the particle can be calculated as the surface integral of the time-averaged second-order pressure \( p_2 \) and momentum flux tensor \( \rho_0 (v_1 v_1) \) at a fixed surface just outside the oscillating sphere, represented by the black circle and green arrows in Fig. 1(b). This follows from the general method of calculating the rate of change of the momentum applied to the inviscid fluid, see Part 1 of the Tutorial Series. The expression for \( F^{rad} \) becomes

\[ F^{rad} = -\int_{\partial \Omega} \left\{ \langle p_2 \rangle \mathbf{n} + \rho_0 \langle (\mathbf{n} \cdot v_1) v_1 \rangle \right\} \]

\[ = -\int_{\partial \Omega} \left\{ \frac{1}{2} \kappa_0 \langle \rho_0^2 \rangle - \frac{1}{2} \rho_0 \langle v_1^2 \rangle \right\} \mathbf{n} \]

\[ + \rho_0 \langle (\mathbf{n} \cdot v_1) v_1 \rangle \}, \]

(10)

As there are no body forces in this problem, any fixed surface \( \partial \Omega \) encompassing the sphere experiences the same force, and given the result of the following scattering theory analysis, it is advantageous to choose a sphere of radius \( r \gg \lambda \) in the far-field region, represented by the dashed circle and the red arrows in Fig. 1(b), with its centre coinciding with that of the spherical particle.

III. Scattering theory

In scattering theory, the scattered field \( \phi_{sc} \) from a point scatterer at the centre of the coordinate system, is represented by a time-retarded multipole expansion. In the far-field region, the monopole and dipole components dominate, \( \phi_{sc} \approx \phi_{mp} + \phi_{dp} \). As verified by insertion into eqn (5), these two components have the form \( \phi_{mp}(r,t) = B(t - r/c_0) \) and \( \phi_{dp}(r,t) = B(t - r/c_0) e^{-i\omega t} \), where \( B \) is a scalar function and \( \mathbf{e} \) a vector function of the retarded argument \( t - r/c_0 \). In first-order scattering theory, \( \phi_{sc} \) must be proportional to the incoming field \( \phi_{in} \). The only physically relevant scalar field is the density, \( \rho \sim \rho_{in} \), while the only relevant vector field is the velocity, \( \mathbf{v} \sim \mathbf{v}_{in} \). Here both \( \rho_{in} \) and \( \mathbf{v}_{in} \) are evaluated at the particle position with time-retarded arguments, and the far-field region \( \phi_{sc} \) must have the form

\[ \phi_{sc}(r,t) = -\frac{f_1 a^3}{3\rho_0} \frac{\partial \rho_{in}(t - r/c_0)}{r} \mathbf{e} \]

\[ -\frac{f_2 a^3}{2} \mathbf{v}_{in}(t - r/c_0) \cdot \mathbf{e}, \]

for \( r \gg \lambda \),

(11)

where the particle radius \( a \), the unperturbed density \( \rho_0 \), and the time derivative \( \partial_t \), are introduced to ensure the correct physical dimension of \( \phi_{sc} \), namely \( \text{m}^2 \text{s}^{-1} \).

The factors 1/3 and 1/2 are inserted for later convenience. The main goal of the calculation is to determine the dimensionless scattering coefficients \( f_1 \) and \( f_2 \).

In the following we use a spherical coordinate system with unit vectors \( (\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z) \) located at the instantaneous centre of the particle. Due to the azimuthal symmetry of the problem, the velocities have no azimuthal component, \( v = v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta \), and all fields depend only on \( r \) and \( \theta \). The polar axis \( \mathbf{e}_z \) points along the instantaneous direction of the incoming velocity \( \mathbf{v}_{in} \), such that \( \mathbf{v}_{in} = v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta \). By the azimuthal symmetry of the problem, the particle must also move in that direction, \( v_r = v_\phi \mathbf{e}_z \).

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A compressible spherical particle (yellow disk) of radius $a$ leading to the outgoing scattered wave $\phi_{sc}$ (red circles and arrows). The resulting first-order wave is $\phi_1 = \phi_{in} + \phi_{sc}$. (b) Sketch of a compressible spherical particle (yellow disk) of radius $a$, compressibility $\kappa_0$, and density $\rho_0$, surrounded by the compressible inviscid bulk fluid (light blue) of compressibility $\kappa_0$ and density $\rho_0$. The fluid is divided into the near-field region for $r \ll \lambda$, with the instantaneous field $\phi_{sc}(t)$, and the far-field region with the time-retarded field $\phi_{sc}(t - r/c_0)$. The radiation force, $F^{\text{rad}}$ (red arrows), evaluated at any point in the far-field region (dashed circle), equals that evaluated at the surface of the sphere (green arrows).

\[ v_{in} = v_{in}e_z = \cos \theta v_{in}e_x - \sin \theta v_{in}e_y \]  
\[ v_{ip} = v_{ip}e_z = \cos \theta v_{ip}e_x - \sin \theta v_{ip}e_y \]

A. Scattering and the radiation force

Before the values of the scattering coefficients are determined, we are able to express the radiation force $F^{\text{rad}}$ in terms of the incoming acoustic wave evaluated at the particle position as well as the coefficients $f_1$ and $f_2$ as follows. When inserting the velocity potentials eqn (9a) and (11) into eqn (10) for $F^{\text{rad}}$, we obtain a sum of terms proportional to the square of $\phi_1 = \phi_{in} + \phi_{sc}$. This results in three types of contributions: (i) squares of $\phi_{sc}$ containing no information about the scattering and therefore yielding zero, (ii) squares of $\phi_{sc}$ proportional to the square of the particle volume $a^6$ and therefore negligible compared to (iii) the mixed products $\phi_{in}\phi_{sc}$ proportional to particle volume $a^4$, and therefore the most dominant contribution to the radiation force. Keeping only these mixed terms, which physically can be interpreted as interference between the incoming and the scattered wave, and using the index notation\(^1\) (including summation of repeated indices), the $i$th component of eqn (10) becomes

\[ F_i^{\text{rad}} = - \int d\Omega \sum_i n_i \left\{ \frac{\epsilon_0^2}{\rho_0} \left( \rho_{in}\rho_{sc} \right) + \rho_0 \left( v_{in}r_{in} \right) - \rho_0 \left( v_{ip}r_{ip} \right) \right\} \delta_{ij} + p_0 \left( v_{in}r_{in} \right) + p_0 \left( v_{ip}r_{ip} \right) \}
\]

\[ + \rho_0 \left( v_{ip}r_{ip} \right) \}
\]

\[ \text{Far field}
\]

\[ \phi_{sc}(t - r/c_0)
\]

\[ \text{Near field \( \phi_{sc}(t) \)}
\]

\[ \text{Inviscid compressible fluid}
\]

Now we see the great advantage of working in the far-field limit. The first term is easily integrated, when appearing in eqn (13), but for the second term we need to get rid of the divergence operator acting on the delta function before we can evaluate the integral. This we manage by Gauss’s theorem. First we note that $\nabla \cdot [v(r)u(r)] = v \nabla \cdot u + u \nabla v$ for any scalar function $v$ and vector function $u$. Therefore, $\int_{\partial \Omega} n \cdot (v u) = \int_{\partial \Omega} d n \cdot (v u) = \int_{\text{int}} d v \cdot (u v)$. We have derived the integral identity $\int_{\text{int}} d s \cdot v = -\int_{\partial \Omega} d n \cdot u + \int_{\partial \Omega} n \cdot (v u)$, and we have obtained using eqn (13) a volume integral encompassing the delta function singularity, thus yielding a non-zero contribution, and a surface integral avoiding the delta function singularity thus yielding zero. Consequently, the resulting expression for $F^{\text{rad}}$ becomes

\[ \sum_{i} \text{d} \phi_{sc} / \rho_{sc} \left( \phi_{sc} - 1/c_0^2 \right) \]

\[ \text{Far field}
\]

\[ \phi_{sc}(t - r/c_0)
\]

\[ \text{Near field \( \phi_{sc}(t) \)}
\]

\[ \text{Inviscid compressible fluid}
\]
The near-field potential

B. The near-field potential

As sketched in Fig. 1(b), the time-retarded argument \( t - r/c_0 \) of the acoustic field \( \phi_1 \) can be replaced by the instantaneous argument \( t \) in the vicinity of the particle of radius \( a \). The reason is that within one oscillation time \( \tau = 2\pi/c_0 \), the retardation time over the distances \( r = a \) is negligible, \( r/c_0 = a/c_0 \ll \lambda/c_0 = \tau \). Therefore, in the near-field region, \( \phi_{\text{ac}} \) derived in eqn (12a) and the scattering potential \( \phi_{\text{sc}} \) of eqn (11) with its monopole and dipole term become

\[
\phi_{\text{ac}}(r) = v_{\text{in}} a \cos \theta, \quad \phi_{\text{sc}}(r, \theta) = \phi_{\text{mon}}(r) + \phi_{\text{dip}}(r, \theta), \quad \text{for } r < \lambda,
\]

where \( \phi_{\text{mon}} \) and \( \phi_{\text{dip}} \) are evaluated at the position of the particle, \( r = 0 \). In first-order scattering theory, the monopole and dipole parts of the problem do not mix; \( f_1 \) is the coefficient in the monopole scattering potential \( \phi_{\text{mon}} \) from a stationary sphere in the incoming density wave \( \rho_{\text{in}} \), while \( f_2 \) is the coefficient in the dipole scattering potential \( \phi_{\text{dip}} \) from an incompressible sphere moving with velocity \( v_p \) in the incoming velocity wave \( v_{\text{in}} \).

C. The monopole coefficient \( f_1 \)

The presence of the particle gives rise to a mass rate \( \partial_m \) of scattered fluid mass given by the first-order, scattered mass flux \( \rho_0 \nabla \phi_{\text{mon}} \). By integration over the surface of the sphere we obtain

\[
\partial_m = \int_{\text{sphere}} \mathbf{v} \cdot \nabla \phi_{\text{mon}} \, d\mathbf{a},
\]

where

\[
f_1 = \frac{4\pi}{3} \nabla \rho_{\text{in}}.
\]

The factor 1/3 was introduced in eqn (11) to make the particle volume \( V_p = (4\pi/3)a^3 \) appear here. The rate of scattered fluid mass can also be written in terms of the rate of change of the incoming density \( \rho_0 + \rho_{\text{in}} \) multiplied by \( V_p \) as \( \partial_m = \partial_m [\rho_0 + \rho_{\text{in}}(t)] V_p(t) \).

D. The dipole coefficient \( f_2 \)

The dipole coefficient \( f_2 \) is related to the translational motion of the particle. For an inviscid fluid, there is only a boundary condition for the radial-direction components of the particle velocity \( v_p \) of eqn (12b) and the dipole part of the fluid velocity, \( e_r v_p = e_r \cdot \nabla (\phi_{\text{mon}} + \phi_{\text{dip}}) \).

At \( r = a \) we obtain \( e_r \cdot \nabla (\phi_{\text{mon}} + \phi_{\text{dip}}) = (1 - f_2) v_{\text{in}} \cos \theta \) from eqn (17a) and (17d), whereby eqn (21) becomes

\[
v_p = (1 - f_2) v_{\text{in}}.
\]

The particle velocity \( v_p \) is also given by Newton’s second law with \( \partial V_p = -i\omega v_p \) and the dipole part \( \rho_0 + \rho_{\text{dip}} \) of the fluid pressure acting on the surface of the sphere,

\[
-i \frac{4\pi}{3} \rho_0 \omega v_p = -2\pi a^2 \int \mathbf{V}(\cos \theta) (\rho_0 + \rho_{\text{dip}}) \cos \theta \, d\mathbf{r}.
\]

From eqn (9c), (17a), and (17d) we obtain \( \rho_0 + \rho_{\text{dip}} = i\rho_0 \omega (\phi_{\text{mon}} + \phi_{\text{dip}}) \),

\[
= i \rho_0 \omega \left[ 1 + \frac{1}{2} f_2 \right] v_{\text{in}} \cos \theta,
\]

which together with eqn (23) leads to

\[
\rho v_p = \left[ 1 + \frac{1}{2} f_2 \right] v_{\text{in}}, \quad \text{with } \rho = \frac{\rho_0}{\rho_0}.
\]

The dipole coefficient \( f_2 \) follows from eqn (22) and (25),

\[
f_2(\hat{\rho}) = \frac{2(\hat{\rho} - 1)}{2\hat{\rho} + 1}.
\]

E. The resulting radiation force

In summary, the resulting radiation force \( \mathbf{F}^{\text{rad}} \) on a small, spherical particle \( (a \ll \lambda) \) in an inviscid fluid is the gradient of an acoustic potential \( U^{\text{rad}} \),

\[
\mathbf{F}^{\text{rad}} = -\nabla U^{\text{rad}},
\]

\[
U^{\text{rad}} = \frac{4\pi}{3} \nabla \left( f_1 \frac{1}{2} \kappa_p (\rho_0^2) - f_2 \frac{3}{2} \rho_0 v_{\text{in}}^2 \right),
\]

\[
f_1(\hat{\kappa}) = 1 - \kappa, \quad \text{with } \kappa = \frac{\kappa_p}{\kappa_0},
\]

\[
f_2(\hat{\rho}) = \frac{2(\hat{\rho} - 1)}{2\hat{\rho} + 1}, \quad \text{with } \hat{\rho} = \frac{\rho_p}{\rho_0}.
\]

IV. Standing plane wave

Our prime example of the acoustic radiation force is the 1D planar standing
The radiation force is found by
\[ F = \frac{p_a}{\rho_0 c_0} \cos(kz) \cos(ot) \] (28a)
\[ p_{\text{rad}}(z,t) = p_c \cos(kz) \sin(ot), \] (28b)
\[ p_{\text{rad}}(z,t) = \frac{p_c}{c_0} \cos(kz) \sin(ot), \] (28c)
\[ v_{\text{rad}}(z,t) = -\frac{p_c}{\rho_0 c_0} \sin(kz) \cos(ot) e_z, \] (28d)
where we have used the usual real-time representation. With these fields, the time averages needed in eqn (27) are simply \( \langle \cos^2(ot) \rangle = \langle \sin^2(ot) \rangle = \frac{1}{2} \), and we arrive at the following expression for the radiation potential
\[ U_{\text{rad}} = \left[ \frac{f_1}{3} \cos^2(kz) - \frac{f_2}{2} \sin^2(kz) \right] \times \pi \nu \kappa_0 p_c^2. \] (29)

The radiation force is found by differentiation,
\[ F_z = -\partial_z U_{\text{rad}} = \frac{4\pi \Phi(k, \rho)ka^3 E_{\text{ac}} \sin(2kz)}{3\pi \eta}, \] (30a)
\[ E_{\text{ac}} = \frac{p_c^2}{4\rho_0 c_0} \] (30b)
\[ \Phi(k, \rho) = \frac{1}{3} f_1(k) + \frac{1}{2} f_2(\rho) \]
\[ = \frac{1}{3} \left[ \frac{5\rho - 2}{2\rho + 1} - k \right]. \] (30c)

Most of the parameters needed as input for theoretical calculations can easily be estimated from table values of materials and from the geometry of the given acoustofluidic device. However, the energy density is not so easy to estimate, since the coupling of acoustic energy from the piezo transducer into the fluidic system is hard to predict as discussed in Part 3 and 4 of the Tutorial Series. A typical value for low-voltage (\( \leq 10 \) V) piezo transducers running at a few MHz on silicon/glass chips is \( 10^18 \) to \( 10^21 \) J m\(^{-3}\). (31)

In the following, we present experimental validation of the above theory of the acoustic radiation force.

V. Acoustophoretic particle tracks

Basic physical properties of acoustophoresis, such as energy density, local pressure amplitudes, acoustophoretic velocity fields, resonance line shapes, and resonance Q factors, are most easily studied in simple rectangular channels embedded in silicon/glass chips like those described in Part 5 of the Tutorial Series. Examples of this approach are given in refs. 13 and 14. In both of these papers, the microfluidic chips under study contain a straight channel with one inlet and one outlet. Typical dimensions of the channels are length \( l = 40 \) mm, width \( w = 0.38 \) mm, and height \( h = 0.16 \) mm. The particles were liquid suspensions of 5 μm-diameter polystyrene microbeads in concentrations ranging from 0.1 g L\(^{-1}\) to 0.5 g L\(^{-1}\). The ultrasound frequency is around 2 MHz corresponding to wavelength \( \lambda \) of 0.75 mm ensuring the validity of the basic assumption \( a \ll \lambda \) of the theory.

In Part 2 of the Tutorial Series we have already reviewed the determination of the acoustic resonance properties in ref. 13, such as the Q factor and the resonance width. Here we will illustrate the experimental validation of the above theory by briefly reviewing the study of the acoustophoretic particle tracks in refs. 13 and 14.

Assuming the channel is aligned with the x-axis and the ultrasonic standing wave is applied in the transverse z-direction, the path of a microbead moving by acoustophoresis is traced out by the time-dependent co-ordinates \( (x(t), z(t)) \). A particularly simple analytical expression for the transverse part, \( z(t) \), of such a path can be obtained from the acoustic radiation force eqn (30a), valid for the 1D planar standing \( \lambda/2 \)-wave of transducers, eqn (28b). For slowly moving micrometre-sized particles we can safely neglect inertial effects and determine the transverse path, \( z(t) \), by balancing the acoustophoretic force \( F_{\text{rad}} \) with the viscous Stokes drag force, \( F_{\text{drag}} = -6\pi \eta v p \), from the quiescent liquid. This force balance results in an expression for the position-dependent particle speed \( v_p \)
\[ v_p(z) = \frac{2\pi \Phi(k, \rho) E_{\text{ac}} \sin(2kz)}{3\pi \eta}. \] (32)

Writing \( v_p = \frac{dz}{dt} \), the resulting differential equation for the transverse particle path \( z(t) \) can be solved analytically by separation in the variables \( z \) and \( t \), and using the integral \( 2[\sin(2s)/\sin(s)] = \ln|\tan(s)| \)
\[ z(t) = \frac{1}{k} \arctan\left\{ \tan[kz(0)] \right\} \]
\[ \times \exp\left\{ \frac{4\Phi}{3(ka)^2} E_{\text{ac}} \right\}, \] (33)

![Fig. 2 A cross-sectional sketch of a straight, hard-walled (green) water-filled channel (blue) of width \( w \) with a transverse, standing, ultrasound \( \lambda/2 \)-pressure resonance \( p_{\text{rad}}(z) = p_c \cos(kz) \) (gray dashed line), \( k = \pi/w \). Relative to \( p_{\text{rad}}(z) \), the radiation force \( F_{\text{rad}}(z) \) on a small suspended particle is period doubled and phase shifted. For a contrast factor \( \Phi > 0 \) we have \( F_{\text{rad}}(z) = \sin(2kz) \) (black arrows), and for \( \Phi < 0 \) we have \( F_{\text{rad}}(z) = -\sin(2kz) \) (red arrows). Consequently, the resulting particle motion is towards and away from the nodal plane (yellow dashed line), respectively.](image-url)
where \( z(0) \) is the transverse position at time \( t = 0 \). In Fig. 3(a) is shown the experimental validation of eqn (33) from ref. 14. The blue points are data for actual particle paths determined by particle tracking frame by frame from a recorded CCD video of the particle motion. The black lines are fitted to data using eqn (33) with only one fitting parameter, the acoustic energy density \( E_{ac} \). The average of the determination of \( E_{ac} \) for 100 particle tracks by this method resulted in 
\[ E_{ac} = (103 \pm 12) \text{ J m}^{-3}. \]

In Fig. 3(b) are shown micro-particle image velocimetry (micro-PIV) measurements of the transverse acoustophoretic velocity \( v_p(z) \). For a microscope field-of-view covering a 0.85-mm-long segment of the 0.38-mm-wide channel, the data are obtained from the first image-pair in 100 repeated experiments of acoustophoretic focusing each starting from a homogeneous particle distribution. After averaging along the channel length, the data are fitted to the acoustophoretic particle velocity \( v_p(z) \) of eqn (32) with \( E_{ac} \) as the only fitting parameter. The resulting value of \( E_{ac} = (98.0 \pm 1.1) \text{ J m}^{-3} \) is in excellent agreement with, and more precise than, the particle tracking method. These results provide a good validation of the theory.

Inverting the expression, we can also calculate the time \( t \) it takes a particle to move from any initial position \( z(0) \) to any final position \( z(t) \),
\[ t = \frac{3\eta}{4\Phi c_0^2 E_{ac}} \ln \left( \frac{\tan[kz(t)]}{\tan[kz(0)]} \right) = \frac{3\eta}{4\Phi u_0^2 c_0^2 E_{ac}} \ln \left( \frac{\tan[kz(t)]}{\tan[kz(0)]} \right). \]  
(34)

This expression is important for designing acousto-fluidic devices to separate particles having the same sign of their acoustophoretic contrast factor \( \Phi \). In this case separation must be based on variances in the time \( t(w) \) it takes a particle to be focused transversely given the width \( w \) of the microfluidic channel. If the axial convection speed of the carrier liquid is \( v_0 \), then the distance \( \Delta t/(v_0, w) \) a given particle has to flow along the channel before it has moved the transverse focus distance \( w \) can be written as
\[ \Delta t/(v_0, w) = v_0 t(w) \propto \kappa^{-1} \Phi^{-1} v_0 u_0^{-2} E_{ac}^{-1}. \]  
(35)

The larger a particle, the shorter it has to be convected before it has been focused. An analysis of the acoustophoretic focus time in terms of the focusing ability is provided in ref. 14.

VI. Energy density as function of the applied piezo voltage

In Part 4 of the Tutorial Series, the basic theory of piezoelectric actuation of ultrasonic resonances in water-filled silicon/glass microchannels is presented. It is shown that a linear relation exists between the applied peak-to-peak voltage \( U_{pp} \) of the piezo transducer responsible for exciting the ultrasonic resonance and the induced acoustic pressure amplitude \( p_a \).

By eqn (30b) \( E_{ac} \) thus scales with the square of \( U_{pp} \),
\[ E_{ac} \propto p_a^2 \propto U_{pp}^2. \]  
(37)

For the energy density \( E_{ac} \approx 100 \text{ J m}^{-3} \), obtained in Fig. 3, we find \( p_a = 1 \text{ MPa} \) or \( 4 \times 10^{-4} \text{ times the cohesive energy density } 2.6 \text{ GPa of water. Equivalently, the density fluctuations are } 4 \times 10^{-4} \text{ times } \rho_0, \]
\[ \rho_0 = 2\sqrt{\rho_0 c_0^2 E_{ac}} = 0.094 \text{ MPa} \sqrt{\frac{E_{ac}}{1 \text{ J m}^{-3}}}. \]  
(38)

This scaling law was tested in ref. 13 by plotting the values of \( E_{ac} \) extracted by the above-mentioned particle-tracking method versus the applied piezo-transducer voltage \( U_{pp} \). The result is shown in Fig. 4 for ten values of \( U_{pp} \) in the range 0.4 V to 1.9 V. A power-law fit resulted in the power 2.07, less than 5% from the expected power of 2.

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The measurements of the acoustic energy density \( E_{ac} \) also allows for a determination of the pressure amplitude \( p_a \), For water at room temperature we obtain from eqn (30b) that
\[ p_a = 2\sqrt{\rho_0 c_0^2 E_{ac}} = 0.094 \text{ MPa} \sqrt{\frac{E_{ac}}{1 \text{ J m}^{-3}}}. \]  
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and thus the acoustic perturbation theory is expected to hold even at resonance.

VII. Viscous corrections to the radiation force

The theory of the acoustic radiation force has so far been developed under the assumption of an inviscid fluid. Going back to the perturbation theory reviewed in Part 2 of the Tutorial Series, this amounts to neglecting the viscous term \( \eta \nabla^2 v_1 \) relative to \( \rho_0 \partial_t v_1 \) in the Navier–Stokes equation. Far from any rigid boundaries, this is a good approximation. However, the bulk velocity oscillating at an ultrasound frequency, \( \omega \), must match the no-slip boundary condition at any given rigid wall, and it is well known by momentum diffusion considerations that during one oscillation, the presence of a wall can be felt up to the penetration depth \( \delta \), given in terms of the kinematic viscosity \( \nu = \eta / \rho_0 \) as,

\[
\delta = \sqrt{\frac{2 \nu}{\omega}} = 0.6 \mu m, \quad (39)
\]

where the value is for 1 MHz ultrasound in water at room temperature. For distances within a few times \( \delta \), large velocity gradients may occur in eqn (2b), such that \( \eta \nabla^2 \delta^2 \gtrsim \rho_0 \partial_t v_1 \), and viscosity cannot be neglected. This viscous fluid layer surrounding a given particle is referred to as the acoustic boundary layer. For a particle radius \( a \gg \delta \) the boundary layer is of negligible relative size, and the inviscid theory is expected to be a good approximation.

In previous works by Doinikov and by Danilov and Mironov, general theoretical schemes for the radiation force have been developed, but analytical expressions were only provided in the special limits of \( a \ll \delta \ll \lambda \) and \( \delta \ll a \ll \lambda \). Given the magnitude of \( \delta \) above, the range of applicability of these published expressions for viscous corrections is severely limited. In recent work by Settnes and Bruus, an analytical expression for the radiation force was derived for any (small) particle size \( \delta, a \ll \lambda \) using the classic Prandtl–Schlichting boundary-layer theory combined with a stream-function formulation of the acoustic boundary layer. The inviscid bulk field is coupled to the motion of the particle through the boundary layer, and not directly as in eqn (21) above.

The result of the analysis of ref. 21 is that the monopole scattering coefficient \( f_1 \) is unchanged (the mass scattering and the compressibility are unaffected by viscosity), while \( f_2 \) becomes complex-valued,

\[
f_2(\bar{\rho}, \delta) = \frac{2[1 - \gamma(\delta)](\bar{\rho} - 1)}{2\bar{\rho} + 1 - 3\gamma(\delta)}, \quad (40a)
\]

\[
\gamma(\delta) = -\frac{3}{2} [1 + i(1 + \delta)] \delta, \quad (40b)
\]

with \( \delta = \frac{\delta}{\bar{a}} \).

The viscosity-dependent correction to the final expression (27) consists in replacing \( f_2(\bar{\rho}) \) by \( Re[f_2(\bar{\rho}, \delta)] \). In the inviscid case \( \delta = 0 \), we find that \( f_2(\bar{\rho}, \delta = 0) = f_2(\bar{\rho}) \), as expected, and for neutral-buoyancy particles (\( \bar{\rho} = 1 \) ) \( f_2 \) is identically zero. As a function of decreasing particle radius \( a \), the value of \( f_2(\bar{\rho}, \delta) \) saturates asymptotically, and for \( \delta f(\bar{\rho}, \delta) \gg 1 \) is \( 2(3)(\bar{\rho} - 1) \).

Using tabulated values of the material parameters, it is found that the relative change in the acoustic contrast factor \( \Phi \) is about 1% or less for 5 μm-diameter, near-neutral buoyancy, polystyrene particles in water, but as much as 25% for pyrex glass particles with a diameter of 0.5 μm.

VIII. Concluding remarks

In this Tutorial Paper we have used first-order and time-averaged second-order perturbation theory to derive an expression for the acoustic radiation force \( F_{rad} \) at wavelength \( \lambda \) for a small spherical particle of radius \( a \) in an inviscid fluid. The expression is the sum of the monopole term for a compressible, stationary particle with coefficient \( f_1 \), which depends only on the relative compressibility \( \tilde{k} \), and the dipole term for a moving, incompressible particle with coefficient \( f_2 \), which depends only on the relative density \( \bar{\rho} \). Examples of experimental validation of the theory have been presented, and a brief review of how the inclusion of the fluid viscosity alters the result. Viscosity plays no role for near-neutral buoyancy particles, while significant corrections appear for particles with a density differing greatly from water. A more detailed treatment of the acoustic boundary layer is given in the coming parts of Tutorial Series treating acoustic streaming.

In this tutorial paper, we have not discussed size effects for particles with radius \( a \) comparable to or larger than the acoustic wavelength \( \lambda \). A good entry to such studies is the theoretical analysis by Hasegawa.

Another aspect not touched upon here, and which also needs more studies is particle–particle interactions. We have only studied the single-particle theory. However, at least two effects play a role as the concentration of the suspended particles is increased. One is hydrodynamic interaction, where one particle feels the Stokes drag from the wake produced by the motion of another particle. A very good and general introduction is given in the textbook by Happel and Brenner, and an explicit example of many-particle effects in microchannel magnetophoresis as function of concentration is given in ref. 24.

Fig. 4 Measured acoustic energy density \( E_{ac} \) versus applied peak-to-peak voltage \( U_{pp} \) on the piezo transducer (points) using the particle-tracking method on 5 μm-diameter polystyrene particles in water. The power law fit (full line) to the data is close to the expected square law, \( E_{ac} \propto U_{pp}^2 \). Adapted from ref. 13.
In the coming years we will likely see more work on acoustofluidics and high particle concentration acoustophoresis and its application to biomedical samples.

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