

Single-particle path integral for composite fermions and the renormalization of the effective mass

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To study composite fermions around an even denominator fraction, we adapt the phase-space single-particle path integral technique for interacting electrons in zero magnetic field developed recently by Golubev and Zaikin [Phys. Rev. B **59**, 9195 (1999)]. This path integral description gives an intuitive picture of composite fermion propagation very similar to the Caldeira-Leggett treatment of a particle interacting with an external environment. We use this description to explain the origin of the famous cancellation between the self-energy and the vertex corrections in semiclassical transport measurements. The effective range of the cancellation is given by the size of the propagating particle, which for the Coulomb interaction scales with the temperature T as $T^{-1/4}|\ln T|^{-1}$ in the semiclassical limit. Using this scheme we find that the effective mass in the semiclassical limit for composite fermions in GaAs is approximately 6 times the bare mass.

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I. INTRODUCTION

In this article we address the question of semiclassical propagation of an interacting system in a real-space formulation.¹ In particular we have in mind the composite fermion (CF) system which is seen in the fractional quantum Hall regime around filling factor $\frac{1}{2}$. The experimental efforts so far indicate that this system is to a high degree a classical system in the sense that it is very hard to observe any kind of interference effects.

In elementary textbooks² the notion of semiclassical transport is sometimes introduced through localized wave packets, and these appear to be essential building blocks. Historically this has really not always been the case. For instance the derivation of the Boltzmann equation in the case of impurity scattering typically only uses individual quasiparticles with a given momentum, i.e., delocalized plane waves.³ This is also the case for other weakly interacting systems like electrons scattering on phonons at low temperatures. However, a Boltzmann equation for electrons scattering on hot phonons could not be derived using plane waves. Prange and Kadanoff solved that problem by shifting focus from individual quasiparticles to deformations of the Fermi surface.⁴ In real space this step corresponds to forming an electron wave packet that is localized in the direction of propagation. This localization in the direction of propagation causes the electron-phonon scattering to be very brief in time due to the huge difference between the Fermi velocity and the sound velocity. Recently, Kim, Lee, and Wen showed that the Boltzmann equation for composite fermions only makes sense for smooth Fermi surface deformations.⁵ In real space the wave packet now also needs to be localized in the direction transverse to the direction of propagation, so the composite fermions is the first system where the textbook introduction actually is a proper description. For composite fermions the necessity of transverse confinement of the wave packet can be understood intuitively. The important interaction is mediated through a gauge field and the coupling is thus through the flux enclosed by an area. For a localized wave packet with a finite width propagating along a classical path the relevant area is the area swept by the wave packet.

The width of the packet is thus a natural cutoff on the transverse gauge fluctuations. In this article we pursue some of the consequences of this idea.

We will study the real-space propagation with the help of phase-space path integrals. Originally, the Feynman path integral technique was developed for noninteracting systems.⁶ Feynman and Vernon⁷ and Caldeira and Leggett⁸ expanded the idea to systems (e.g., a particle) interacting with an environment. The result of integrating out the environment was an extra definite contribution to the effective action in the path integral. Recently, Golubev and Zaikin extended the technique to cover the linear response regime for the reduced one-particle density matrix in a Coulomb interacting gas of electrons.¹ Their main insight was that when focusing on a single particle (response) the interaction with all the other particles can be viewed as an effective environment in the Caldeira-Leggett sense. The Fermi statistics reveals itself only in the complicated form of the resulting effective action, which depends on the distribution function.

For the CF system the Chern-Simons transformation introduces the Chern-Simons gauge field as the effective environment. The resulting noninteracting particles we call composite fermions. The Chern-Simons transformation only attaches two flux quanta to each electron. Recently, it has been suggested that a better description of the quasiparticles is obtained by associating two vortices to each electron resulting in neutral quasiparticles with a momentum-dependent dipole moment.⁹⁻¹¹ In the Chern-Simons description the formation of the neutral quasiparticles is presumably due to dynamical screening. Here we want to stress that the one-particle density matrix in our approach is defined in terms of the original electron variables, so the connection between the quasiparticles and the physical response is automatically included. The Chern-Simons transformation is only used to introduce an effective environment in a relatively simple way. Another scheme might lead to a different physical picture and eventually suggest other approximations.

Anyway, within the resulting single-particle phase-space path integral formulation we show how to define the effective mass in the semiclassical limit and calculate it to be 0.4 times the vacuum electron mass in a typical experiment.¹²

This article is organized as follows. In Sec. II we adapt the Golubev-Zaikin extension of the Caldeira-Leggett methodology to the case of composite fermions. We summarize the main points of the resulting path integral technique in Sec. II G. In Sec. III the path integral technique is on a formal level compared with the Kubo formula description of transport and, more briefly, with the Landauer-Büttiker scattering approach to transport. In Sec. IV we discuss the typical semiclassical approximation scheme of saying that the two paths are almost identical, and the typical width between the two paths is calculated. In Sec. V the calculation of the effective mass of composite fermions is carried through in the semiclassical limit.

We have found it necessary to introduce quite a few averages and shorthand notation. As a reader's guide we list them here. $\langle X \rangle$ is the full quantum and thermal average. A subscript 0 as in $\langle X \rangle_0$ means that the average is performed for the noninteracting electron gas. $\langle X \rangle_{a,t}$ is the average over different Chern-Simons gauge field configurations; the precise definition is Eq. (17). $\langle |X| \rangle = \langle |X| \rangle_{[0,t]}$ in Eq. (68) is the average in the time interval $[0,t]$ over different single-particle paths, where each path is given a weight that reflects its importance in a given experiment. It thus depends on what experiment is considered. $\langle\langle X \rangle\rangle = \langle\langle X \rangle\rangle_{[0,t]}$ is the shorthand notation for the sum over single-particle paths including the weight due to the interactions. It is defined in Eq. (69). $\langle\langle\langle X \rangle\rangle\rangle$ defined in Eq. (A4) is a shorthand notation for the Grassmann integral. It is only used in Appendix A. Quite generally the superscript "eq" refers to the average being calculated in equilibrium. For instance $\langle X \rangle_{a,t}^{\text{eq}}$ in Eq. (27) is the equilibrium version of $\langle X \rangle_{a,t}$.

II. DENSITY MATRIX

We consider a two-dimensional electron gas in a static and homogeneous external magnetic field B_{ext} perpendicular to the plane. We write B_{ext} as a scalar since its direction is fixed, and likewise we consider the rotation of any vector potential in the two-dimensional (2D) plane as a scalar. Furthermore, we also apply an external electrical potential $\phi(\mathbf{r},t)$. The external potentials are written in four-vector form as $A = (\phi, \mathbf{A})$. In this section the goal is to derive a path integral description of the linear response of the reduced one-particle density matrix ρ to the potential $\phi(\mathbf{r},t)$. For simplicity we will in this section neglect the influence of impurities, but it is straightforward to include them in the formalism as an extra external potential.

Our notation is introduced in Sec. II A. In Sec. II B the Chern-Simons transformation to composite fermions is carried out. This renders the problem in the Caldeira-Leggett form of a particle interacting with an environment. The important formula is Eq. (16). The properties of the environment, which in our case is the gauge field, are discussed in Sec. II C. The density matrix in the presence of this environment fulfills a nonlinear differential equation, presented in Sec. II D, and in Sec. II E the linear response is discussed on the differential equation level. In Sec. II F the linearized differential equation is solved formally with the help of single-particle path integrals and the environment is integrated out.

This leaves us with a single-particle path integral formulation of the linear response for composite fermions. In Sec. II G the major results are summarized.

A. Formulation of the problem using electrons as the fundamental particles

The system is described by the Hamiltonian

$$H_{\text{el}} = H_0[A] + H_{\text{int}}. \quad (1)$$

Here

$$H_0[A] = \int d\mathbf{r} \times \hat{\psi}^\dagger(\mathbf{r}) \left\{ \frac{1}{2m} \left[\frac{\hbar}{i} \nabla + e\mathbf{A}(\mathbf{r}) \right]^2 - e\phi(\mathbf{r},t) - \mu \right\} \hat{\psi}(\mathbf{r}) \quad (2)$$

is the Hamiltonian for noninteracting particles in the 2D plane represented by the second-quantized electron field operators $\hat{\psi}(\mathbf{r})$ and $\hat{\psi}^\dagger(\mathbf{r})$, while

$$H_{\text{int}} = \frac{e^2}{2} \int d\mathbf{r} \int d\mathbf{r}' \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}^\dagger(\mathbf{r}') v(\mathbf{r} - \mathbf{r}') \hat{\psi}(\mathbf{r}') \hat{\psi}(\mathbf{r}) + H_{\text{eb}} \quad (3)$$

is the part due to electron-electron interaction. H_{eb} represents the interaction between the electrons and a fixed neutralizing positively charged background jellium. m is the mass of the electron, μ is the chemical potential, $-e$ is the electron charge, and $e^2 v(\mathbf{r})$ is the Coulomb pair interaction between electrons. If the screening effect of the back gate is neglected, $v(\mathbf{r}) = 1/(4\pi\epsilon|\mathbf{r}|)$, where ϵ is the permittivity of the medium surrounding the two-dimensional electron gas.

All the physical one-particle quantities of the system, like the current and the density, can be expressed in terms of the reduced one-particle density matrix

$$\rho_c(\mathbf{r}, \mathbf{r}'; t) = \langle \hat{\psi}^\dagger(\mathbf{r}', t) \hat{\psi}(\mathbf{r}, t) \rangle. \quad (4)$$

Here $\langle X \rangle$ denotes the full quantum mechanical and thermal average of X . The subscript "c" stands for canonical, because upon a Fourier transformation the annihilation and creation operators correspond to states labeled by the canonical momentum. This labeling is not gauge invariant, so neither is $\rho_c(\mathbf{r}, \mathbf{r}')$, but it has some nice formal properties we will make use of in the following. Final physical expectation values we prefer to phrase in terms of the gauge invariant density matrix

$$\rho(\mathbf{r}, \mathbf{r}') = \rho_c(\mathbf{r}, \mathbf{r}') \exp\left(i \frac{e}{\hbar} \int_{\mathbf{r}'}^{\mathbf{r}} d\mathbf{r}'' \cdot \mathbf{A}(\mathbf{r}'') \right), \quad (5)$$

where the path of integration is chosen to be along the straight line from \mathbf{r}' to \mathbf{r} . In general different choices of the integration path lead to different values of ρ , but we are mainly interested in local probes like the density and the current. Any path that locally approaches a straight line then yields the same result. In such local probes it is convenient to introduce variables for the average $\mathbf{R} = \frac{1}{2}(\mathbf{r} + \mathbf{r}')$ and for the difference $\Delta\mathbf{r} = \mathbf{r} - \mathbf{r}'$. We use a new symbol for the density matrix expressed in these variables:

$$w(\mathbf{R}, \Delta \mathbf{r}) = \rho \left(\mathbf{R} + \frac{\Delta \mathbf{r}}{2}, \mathbf{R} - \frac{\Delta \mathbf{r}}{2} \right). \quad (6)$$

The Fourier transform with respect to the difference variable is the corresponding gauge-invariant Wigner distribution function

$$w(\mathbf{R}, \mathbf{p}) = \int d\Delta \mathbf{r} \exp \left(-\frac{i}{\hbar} \mathbf{p} \cdot \Delta \mathbf{r} \right) w(\mathbf{R}, \Delta \mathbf{r}) \xrightarrow{\text{eq. non-int.}} f_{\text{F}}(\epsilon_{\mathbf{p}}), \quad (7)$$

where the arrow denotes that in equilibrium for a noninteracting system w reduces to the Fermi distribution function $f_{\text{F}}(\epsilon) = \{1 + \exp[(\epsilon - \mu)/(k_{\text{B}}T)]\}^{-1}$. $\epsilon_{\mathbf{p}}$ is the single-particle energy.

The thermal and quantum average of ρ_c and ρ can be calculated with the help of a path integral over Grassmann fields $\bar{\psi}_{\text{el}}$ and ψ_{el} . $\bar{\psi}_{\text{el}}$ and ψ_{el} correspond to the operators $\hat{\psi}^\dagger$ and $\hat{\psi}$, and they should not be confused. Using this representation we obtain

$$\rho(\mathbf{r}, \mathbf{r}'; t) = \exp \left(i \frac{e}{\hbar} \int_{\mathbf{r}'}^{\mathbf{r}} d\mathbf{r}'' \cdot \mathbf{A}(\mathbf{r}'') \right) \frac{\int \mathcal{D}\bar{\psi}_{\text{el}} \int \mathcal{D}\psi_{\text{el}} \bar{\psi}_{\text{el}}(\mathbf{r}', t) \psi_{\text{el}}(\mathbf{r}, t) \exp \left(\frac{i}{\hbar} S[\bar{\psi}_{\text{el}}, \psi_{\text{el}}; t] \right)}{\int \mathcal{D}\bar{\psi}_{\text{el}} \int \mathcal{D}\psi_{\text{el}} \exp \left(\frac{i}{\hbar} S[\bar{\psi}_{\text{el}}, \psi_{\text{el}}; t] \right)}, \quad (8)$$

where the action S is given by

$$S[\bar{\psi}_{\text{el}}, \psi_{\text{el}}; t] = \int_{C_t} d\sigma \int d\mathbf{r} \bar{\psi}_{\text{el}}(\mathbf{r}, \sigma) i\hbar \frac{\partial}{\partial \sigma} \psi_{\text{el}}(\mathbf{r}, \sigma) - \int_{C_t} d\sigma H_{\text{el}}[\bar{\psi}_{\text{el}}, \psi_{\text{el}}]. \quad (9)$$

The contour C_t runs from minus infinity up to the observation time t and back again to minus infinity. The contour time $\sigma = (s, \mu)$, where s is a real time and $\mu = \pm$ is a contour index that takes the value $+$ on the forward part of the contour and $-$ on the backward in time part of the contour; see Fig. 1. In the prefactor $\bar{\psi}_{\text{el}}(\mathbf{r}', t) \psi_{\text{el}}(\mathbf{r}, t)$ the creation Grassmann field $\bar{\psi}_{\text{el}}(\mathbf{r}', t)$ has to come later than the annihilation Grassmann field $\psi_{\text{el}}(\mathbf{r}, t)$ on the contour C_t , but whether both fields are on the forward or backward parts of the contour C_t or $\psi_{\text{el}}(\mathbf{r}, t)$ is on the forward part, while $\bar{\psi}_{\text{el}}(\mathbf{r}', t)$ is on the backward in time part of C_t is immaterial.

B. Chern-Simons field-dependent density matrix

The first step towards obtaining an effective single-particle path integral description is to introduce fluctuating fields as extra integration variables such that the action be-

comes quadratic in the Grassmann fields. The electron physics is then reduced to that of noninteracting electrons in a fluctuating field background. In the case of zero magnetic field the relevant transformation is a Hubbard-Stratonovich transformation that introduces an effective electric potential at each contour point.¹ In the case of composite fermions the relevant transformation is a Chern-Simons transformation that introduces a Chern-Simons gauge field $a = (a_0, \mathbf{a})$ at each contour point and that also performs a singular gauge transformation on the Grassmann fields attaching two magnetic flux quanta h/e to each electron. The transformed Grassmann fields we denote by $\bar{\psi}$ and ψ , not to be confused with $\hat{\psi}^\dagger$, $\hat{\psi}$, $\bar{\psi}_{\text{el}}$, and ψ_{el} .

The action transforms into the standard action for composite fermions:¹³

$$S_{\text{CF}}[\bar{\psi}, \psi, a, A_{\text{eff}} - a; t] = S_{\text{CS}}[a; t] + e \int_{C_t} d\sigma \int d\mathbf{r} a_0(\mathbf{r}, \sigma) n + \int_{C_t} d\sigma \int d\mathbf{r} \bar{\psi}(\mathbf{r}, \sigma) i\hbar \frac{\partial}{\partial \sigma} \psi(\mathbf{r}, \sigma) - \int_{C_t} d\sigma H_0[A_{\text{eff}} - a]. \quad (10)$$

Here $S_{\text{CS}}[a; t]$ is the Chern-Simons action

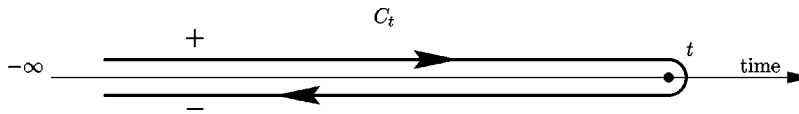


FIG. 1. The Grassmann fields $\bar{\psi}_{\text{el}}$, ψ_{el} , $\bar{\psi}$, and ψ and the Chern-Simons gauge field $a = (a_0, \mathbf{a})$ on the contour C_t . As can be seen from the drawing the contour C_t runs along the physical time axis from $-\infty$ up to time t and then back again to $-\infty$. The contour time σ is on the forward and backward parts of the contour denoted $\sigma = (s, +)$ and $\sigma = (s, -)$, respectively, where s is the physical time. To each physical time $s < t$ there is thus as an example two independent Chern-Simons gauge fields $a(s, +)$ and $a(s, -)$. If in particular $t = \infty$, the usual Keldysh contour C_∞ is recovered.

$$\begin{aligned}
S_{\text{CS}}[a;t] &= \int_{C_t} d\sigma \int d\mathbf{r} \\
&\times \left(\frac{e^2}{4\pi\hbar} a_0(\sigma) \nabla \times \mathbf{a}(\sigma) + \frac{e^2}{8\pi\hbar} \mathbf{a}(\sigma) \times \frac{\partial \mathbf{a}}{\partial \sigma} \right) \\
&- \frac{e^4}{2(4\pi\hbar)^2} \int_{C_t} d\sigma \int d\mathbf{r} \int d\mathbf{r}' \\
&\times [\nabla \times \mathbf{a}(\mathbf{r}, \sigma)] v(\mathbf{r} - \mathbf{r}') [\nabla \times \mathbf{a}(\mathbf{r}', \sigma)]. \quad (11)
\end{aligned}$$

The average mean-field magnetic field $2nh/e$ has been subtracted from both the externally magnetic field $\nabla \times \mathbf{A}$ and the Chern-Simons gauge field. The remaining fields are denoted by $A_{\text{eff}} = (0, \mathbf{A}_{\text{eff}}) = (0, \mathbf{A} - (\nabla \times)^{-1} 2(h/e)n)$ and $a = (a_0, \mathbf{a})$, respectively. For notational consistency with Sec. II E treating the linear response regime we have chosen to have a vanishing external potential $\phi=0$. If nonzero, it would be added to A_{eff} . The last term in S_{CS} originates from the interaction term H_{int} , Eq. (3). For later use we notice that in the terms that couple the Chern-Simons gauge field a to the Chern-Simons fields $\bar{\psi}$ and ψ , a only enters in the combination $A_{\text{eff}} - a$.

The prefactor in Eq. (8) transforms as

$$\begin{aligned}
\bar{\psi}_{\text{el}}(\mathbf{r}', t) \psi_{\text{el}}(\mathbf{r}, t) \exp\left(i \frac{e}{\hbar} \int_{\mathbf{r}'}^{\mathbf{r}} d\mathbf{r}'' \cdot \mathbf{A}(\mathbf{r}'') \right) &\leftrightarrow \bar{\psi}(\mathbf{r}', t) \psi(\mathbf{r}, t) \\
&\times \exp\left(i \frac{e}{\hbar} \int_{\mathbf{r}'}^{\mathbf{r}} d\mathbf{r}'' \cdot [A_{\text{eff}}(\mathbf{r}'', t) - \bar{\mathbf{a}}(\mathbf{r}'', t)] \right). \quad (12)
\end{aligned}$$

The average field

$$\bar{\mathbf{a}}(\mathbf{r}'', t) = \frac{1}{2} [\mathbf{a}(\mathbf{r}'', t, +) + \mathbf{a}(\mathbf{r}'', t, -)] \quad (13)$$

is chosen for notational consistency. It is allowed since the a field is continuous on the contour C_t , so it does not matter whether the Chern-Simons field at the turning point t is evaluated on the forward or on the backward part of the contour. The total expression for the density matrix in terms of the transformed fields is

$$\begin{aligned}
\rho(\mathbf{r}, \mathbf{r}'; t) &= \left[\int Da \int D\bar{\psi} \int D\psi \exp\left(\frac{i}{\hbar} S_{\text{CF}} \right) \right]^{-1} \\
&\times \int Da \int D\bar{\psi} \int D\psi \bar{\psi}(\mathbf{r}', t) \psi(\mathbf{r}, t) \\
&\times \exp\left(\frac{i}{\hbar} S_{\text{CF}} + i \frac{e}{\hbar} \int_{\mathbf{r}'}^{\mathbf{r}} d\mathbf{r}'' \cdot [A_{\text{eff}}(\mathbf{r}'') - \bar{\mathbf{a}}(\mathbf{r}'', t)] \right). \quad (14)
\end{aligned}$$

The exponent is quadratic in the Grassmann CF variables $\bar{\psi}$ and ψ and thus describes noninteracting particles in a gauge field history $A_{\text{eff}} - a$. However, the gauge field is not physically realizable, since it is different on the forward and back-

ward contours, $a(s, +) \neq a(s, -)$. Despite this we follow Golubev and Zaikin¹ and define the Chern-Simons field-dependent density matrix

$$\begin{aligned}
\rho_{ac}(A_{\text{eff}} - a; \mathbf{r}, \mathbf{r}', t) &= \left[\int D\bar{\psi} \int D\psi \exp\left(\frac{i}{\hbar} S_{\text{CF}} \right) \right]^{-1} \\
&\int D\bar{\psi} \int D\psi \bar{\psi}(\mathbf{r}', t) \psi(\mathbf{r}, t) \exp\left(\frac{i}{\hbar} S_{\text{CF}} \right), \quad (15)
\end{aligned}$$

in analogy with the usual definition of a density matrix Eq. (4). Notice that the right-hand side only depends on the gauge fields in the combination $A_{\text{eff}} - a$, since the contribution from $S_{\text{CS}}[a;t]$ to the action S_{CF} cancels out. The subscript a on ρ_{ac} is to remind us that ρ_{ac} depends on the field history, while the subscript c stands for canonical in analogy with ρ_c ; see Eq. (4). The physical density matrix is recovered from $\rho_{ac}(A_{\text{eff}} - a; \mathbf{r}, \mathbf{r}', t)$ by averaging over the different a -field histories

$$\begin{aligned}
\rho(\mathbf{r}, \mathbf{r}', t) &= \left\langle \rho_{ac}(A_{\text{eff}} - a; \mathbf{r}, \mathbf{r}', t) \right. \\
&\times \exp\left\{ i \frac{e}{\hbar} \int_{\mathbf{r}'}^{\mathbf{r}} d\mathbf{r}'' \cdot [\mathbf{A}_{\text{eff}}(\mathbf{r}'') - \bar{\mathbf{a}}(\mathbf{r}'', t)] \right\} \Bigg\rangle_{a,t}. \quad (16)
\end{aligned}$$

In the average

$$\langle X \rangle_{a,t} = \frac{\int Da X \exp\left(\frac{i}{\hbar} S_{\text{eff}}[a;t] \right)}{\int Da \exp\left(\frac{i}{\hbar} S_{\text{eff}}[a;t] \right)}, \quad (17)$$

the weight

$$\begin{aligned}
\exp\left(\frac{i}{\hbar} S_{\text{eff}}[a;t] \right) &= \int D\bar{\psi} \int D\psi \\
&\times \exp\left(\frac{i}{\hbar} S_{\text{CF}}[\bar{\psi}, \psi, a, A_{\text{eff}} - a; t] \right) \quad (18)
\end{aligned}$$

is the denominator in Eq. (15). The effective action $S_{\text{eff}}[a;t]$ for the a field also depends on the external gauge field A_{eff} , but this dependence has been suppressed in the notation, since it is not needed in the following.

Equation (16) is an important formula, since it establishes the connection between the density matrix ρ_{ac} depending on a given history of the gauge field and the physically accessible density matrix ρ . Notice that Eq. (16) is written in the Caldeira-Leggett spirit as a single-particle object ρ_{ac} that couples to a fluctuating field that eventually has to be integrated out. As in standard Caldeira-Leggett theory the gauge field is different on the two contours. Below it become clear that, as usual, the average field $\bar{\mathbf{a}}(\mathbf{r}'', t)$ in many respects can be interpreted as an external stochastic physical field. Only keeping the coupling to the average field is thus identical to describing the composite fermions as free particles coupled to the physical gauge field $\bar{\mathbf{a}}$ whose fluctuations are given by the fluctuation-dissipation theorem and the spectral function

for the composite fermion gauge field.¹⁴ For our discussion of the effective mass it is, however, crucial that the two fields are different.

C. Effective action for the a field

Introducing the a -dependent current $\mathbf{j}_a(A_{\text{eff}}-a; \mathbf{r}, s)$ at time s ,

$$\mathbf{j}_a(A_{\text{eff}}-a; \mathbf{r}, s) = \frac{-e}{2m} \left[\frac{\hbar}{i} \frac{\partial}{\partial \mathbf{r}} + e\mathbf{A}_{\text{eff}}(\mathbf{r}) - e\bar{\mathbf{a}}(\mathbf{r}) - \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{r}'} + e\mathbf{A}_{\text{eff}}(\mathbf{r}') - e\bar{\mathbf{a}}(\mathbf{r}') \right] \rho_{ac}(\mathbf{r}, \mathbf{r}'; s) \Big|_{\mathbf{r}'=\mathbf{r}}, \quad (19)$$

it is shown in Appendix A that the effective single-particle action $S_{\text{eff}}[a; t]$ is given by

$$S_{\text{eff}}[a; t] = S_{\text{CS}}[a; t] - \int_{-\infty}^t ds \int d\mathbf{r} \times [e\Delta a_0(\mathbf{r}, s)(\rho_{ac}(\mathbf{r}, \mathbf{r}; s) - n) + \Delta \mathbf{a}(\mathbf{r}, s) \cdot \mathbf{j}_a(\mathbf{r}, s)], \quad (20)$$

where

$$\bar{\bar{R}}(\mathbf{r}', s'; \mathbf{r}'', s'') \equiv -\frac{i}{\hbar} \begin{pmatrix} \langle [-e\rho_{ac}(\mathbf{r}', s'), -e\rho_{ac}(\mathbf{r}'', s'')] \rangle_0 & -\langle [-e\rho_{ac}(\mathbf{r}', s'), \mathbf{j}_a(\mathbf{r}'', s'')] \rangle_0 \\ -\langle [\mathbf{j}_a(\mathbf{r}', s'), -e\rho_{ac}(\mathbf{r}'', s'')] \rangle_0 & \langle [\mathbf{j}_a(\mathbf{r}', s'), \mathbf{j}_a(\mathbf{r}'', s'')] \rangle_0 \end{pmatrix} \Theta(s' - s'') + \begin{pmatrix} 0 & 0 \\ 0 & \frac{e^2}{m} \rho_{ac}(\mathbf{r}', s') \delta(\mathbf{r}' - \mathbf{r}'') \delta(s' - s'') \end{pmatrix}, \quad (23)$$

and the so-called fluctuation kernel $(\bar{\bar{F}})_{\alpha\beta}$ by

$$\bar{\bar{F}}(\mathbf{r}_1, s_1; \mathbf{r}_2, s_2) \equiv \frac{1}{2\hbar} \begin{pmatrix} \langle \{-e\rho_{ac}(\mathbf{r}', s'), -e\rho_{ac}(\mathbf{r}'', s'')\} \rangle_0 & -\langle \{-e\rho_{ac}(\mathbf{r}', s'), \mathbf{j}_a(\mathbf{r}'', s'')\} \rangle_0 \\ -\langle \{\mathbf{j}_a(\mathbf{r}', s'), -e\rho_{ac}(\mathbf{r}'', s'')\} \rangle_0 & \langle \{\mathbf{j}_a(\mathbf{r}', s'), \mathbf{j}_a(\mathbf{r}'', s'')\} \rangle_0 \end{pmatrix}. \quad (24)$$

$\langle X \rangle_0$ means the average over the noninteracting electron gas without the Chern-Simons gauge field a , and Θ is the step function. Below we are mainly going to need the retarded gauge field propagator $D_{\alpha\beta}$ and the correlation function $C_{\alpha\beta}$ for fluctuations of the field defined by

$$D_{\alpha\beta}(\mathbf{r}', s'; \mathbf{r}'', s'') \equiv -\frac{i}{\hbar} e^2 \langle \bar{a}_\alpha(\mathbf{r}', s') \Delta a_\beta(\mathbf{r}'', s'') \rangle_{a,t}^{\text{eq}} = -\frac{i}{\hbar} \Theta(s' - s'') e^2 \langle [a_\alpha(\mathbf{r}', s'), a_\beta(\mathbf{r}'', s'')] \rangle_0, \quad (25)$$

$$\Delta a_\alpha(\mathbf{r}, s) = a_\alpha(\mathbf{r}, s, +) - a_\alpha(\mathbf{r}, s, -) \quad (21)$$

is the difference of the Chern-Simons gauge field on the two parts of C_t . $S_{\text{CS}}[a; t]$ is the usual Chern-Simons action; see Eq. (11). In the notation the explicit dependence of the density matrix ρ_{ac} and the current \mathbf{j}_a on the gauge field $A_{\text{eff}}-a$ has been suppressed. In equilibrium, which is denoted by the superscript “eq,” we have to second order in the a field

$$S_{\text{eff}}^{\text{eq}}[a; t] \approx S_{\text{CS}}[a; t] - \sum_{\alpha, \beta} \int_{-\infty}^t ds' \int_{-\infty}^{s'} ds'' \int d\mathbf{r}' \int d\mathbf{r}'' \times \Delta a_\alpha(\mathbf{r}', s') R_{\alpha\beta}(\mathbf{r}', s'; \mathbf{r}'', s'') \bar{a}_\beta(\mathbf{r}'', s'') + \frac{i}{2} \sum_{\alpha, \beta} \int_{-\infty}^t ds' \int_{-\infty}^{s'} ds'' \int d\mathbf{r}' \int d\mathbf{r}'' \times \Delta a_\alpha(\mathbf{r}', s') F_{\alpha\beta}(\mathbf{r}', s'; \mathbf{r}'', s'') \Delta a_\beta(\mathbf{r}'', s''). \quad (22)$$

The summation variables $\alpha, \beta \in \{0, x, y\}$ run over the single timelike and the two spatial indices, e.g., $a_\alpha = (a_0, a_x, a_y)$. The terms linear in a vanish, since $\rho_{ac}|_{a=0} = n$ and $\mathbf{j}_a|_{a=0} = \mathbf{0}$. The retarded 3×3 response matrix $(\bar{\bar{R}})_{\alpha\beta}$ is given by the usual Kubo formula apart from an extra minus in the part that mixes the density with the currents,

$$C_{\alpha\beta}(\mathbf{r}', s'; \mathbf{r}'', s'') \equiv \frac{e^2}{\hbar} \langle \bar{a}_\alpha(\mathbf{r}', s') \bar{a}_\beta(\mathbf{r}'', s'') \rangle_{a,t}^{\text{eq}} = \frac{e^2}{\hbar} \frac{1}{2} \langle \{a_\alpha(\mathbf{r}', s'), a_\beta(\mathbf{r}'', s'')\} \rangle_0, \quad (26)$$

while the autocorrelation function for Δa_α vanishes: $\langle \Delta a_\alpha(\mathbf{r}', s') \Delta a_\beta(\mathbf{r}'', s'') \rangle_{a,t}^{\text{eq}} = 0$. The average

$$\langle X \rangle_{a,t}^{\text{eq}} = \frac{\int \mathcal{D}a X \exp\left(\frac{i}{\hbar} S_{\text{eff}}^{\text{eq}}[a; t]\right)}{\int \mathcal{D}a \exp\left(\frac{i}{\hbar} S_{\text{eff}}^{\text{eq}}[a; t]\right)} \quad (27)$$

is the average $\langle X \rangle_{a,t}$ of Eq. (17), calculated in equilibrium.

In the Coulomb gauge the most troublesome contributions arise from the low-frequency limit of the so-called transverse part of the gauge field propagator $D_{\perp\perp}$,^{15,16} which in Fourier space is given by

$$D_{\perp\perp}(\mathbf{q}, \omega) = \sum_{ijkl} \varepsilon^{ik} \varepsilon^{jl} \frac{q_i q_j}{q^2} D_{kl}(\mathbf{q}, \omega). \quad (28)$$

Here $\varepsilon^{xx} = \varepsilon^{yy} = 0$, $\varepsilon^{xy} = -\varepsilon^{yx} = 1$, and the indices i, j, k , and l run over x and y . In equilibrium in the limit $\hbar\omega \ll qv_F$ the propagator $D_{\perp\perp}$ is given by the standard approximation¹⁶

$$\begin{aligned} D_{\perp\perp}(\mathbf{q}, \omega) &\approx 2\pi\hbar \frac{q}{p_F} \frac{1}{i\omega - \frac{2\pi e^2 v(q) q^3}{(4\pi\hbar)^2 \hbar p_F}} \\ &= 2\pi\hbar \frac{q}{p_F} \frac{1}{i\omega - g_\eta q^{\eta+1}}. \end{aligned} \quad (29)$$

For the Coulomb interaction $v(q) = \hbar/2\varepsilon q$, so $\eta = 1$ and $g_1 = (1/\hbar p_F^2)^{1/2} \sqrt{\pi} \varepsilon_C$ in terms of the average Coulomb energy $\varepsilon_C = (e^2/4\pi\varepsilon) \sqrt{n}$. n is the density of the electron gas. Other values of $1 \leq \eta < 2$ corresponds to a power law decay $r^{-\eta}$.

In equilibrium \bar{C} is determined from the fluctuation-dissipation theorem. In particular in the Coulomb gauge is the most singular contribution due to the transverse part of the propagator:¹⁴

$$C_{ij}(\mathbf{q}, \omega) \approx - \left(\delta_{ij} - \frac{q_i q_j}{q^2} \right) \coth \left(\frac{\hbar\omega}{2k_B T} \right) \text{Im} D_{\perp\perp}(\mathbf{q}, \omega). \quad (30)$$

D. Equation of motion for the gauge-field-dependent density matrix

In Appendix A we have by brute force differentiation showed that ρ_{ac} fulfills the all important nonlinear equation of motion

$$i\hbar \partial_t \rho_{ac} = (1 - \rho_{ac}) H(t, +) \rho_{ac} - \rho_{ac} H(t, -) (1 - \rho_{ac}). \quad (31)$$

This equation should be read as an operator identity. In Eq. (A12) it is written out. $H(+)$ is the kernel in the quadratic action for the Grassmann CF variables on the forward part of the contour, i.e.,

$$\begin{aligned} H(\mathbf{r}', \mathbf{r}''; t, +) &= \delta(\mathbf{r}' - \mathbf{r}'') \left\{ -e\phi(\mathbf{r}'', t) + e a_0(\mathbf{r}'', t, +) \right. \\ &\quad \left. + \frac{1}{2m} \left[\frac{\hbar}{i} \nabla'' + e \mathbf{A}_{\text{eff}}(\mathbf{r}'') - e \mathbf{a}(\mathbf{r}'', t, +) \right]^2 \right\}. \end{aligned} \quad (32)$$

$a(+)$ $= [a_0(+), \mathbf{a}(+)]$ is the gauge field on the forward contour. Likewise is $H(-)$ the kernel on the backward in time part of the contour. If the gauge field is identical on the forward and backward sections of the contour, Eq. (31)

specializes to the usual equation of motion for the density matrix, $i\hbar \partial_t \rho = [H, \rho]$. The extra complication in the general case is the term quadratic in the density matrix: $-\rho_{ac} [H(+)-H(-)] \rho_{ac}$. However, we restrict ourselves to the linear response regime.

E. Formulation of linear response

We are interested in the response of the system to an external driving field ϕ and in particular in the change in the density matrix: $\delta\rho(\mathbf{r}, \mathbf{r}'; t) = \rho(\mathbf{r}, \mathbf{r}'; t) - \rho^{\text{eq}}(\mathbf{r}, \mathbf{r}'; t)$. The superscript ‘‘eq’’ still denotes equilibrium where $\phi = 0$. In general both the gauge-field-dependent density matrix $\rho_{ac}(A_{\text{eff}} - a; \mathbf{r}, \mathbf{r}', t)$ and the effective action for the Chern-Simons field $S_{\text{eff}}[a; t]$ depend on ϕ , giving rise to two different contributions to $\delta\rho(\mathbf{r}, \mathbf{r}', t)$ —compare with Eqs. (16) and (17). The contribution coming from the variation of $S_{\text{eff}}[a; t]$ can be accounted for by a change of variable from the externally applied field ϕ to the total gauge field felt by the composite fermions:

$$A_{\text{tot}}(a; t) = (\phi(t), \mathbf{0}) + A_{\text{ind}}(a; t). \quad (33)$$

Here A_{ind} is the self-consistently induced gauge field due to the change in the current and the density when ϕ is applied. Technically it is defined by the demand that

$$S_{\text{eff}}[a - A_{\text{ind}}(a); t] = S_{\text{eff}}^{\text{eq}}[a; t]. \quad (34)$$

From this definition it is not immediately clear that the induced field is the same on the forward and backward parts of the contour and thus can be interpreted as a physical field. However, a careful inspection of Eqs. (11) and (20) shows this is the case. Furthermore, it is seen that the induced electromagnetic fields corresponding to A_{ind} are

$$B_{\text{ind}}(A_{\text{eff}} - a; \mathbf{r}, t) = - \frac{4\pi\hbar}{e} \delta\rho_{ac}(A_{\text{eff}} - a; \mathbf{r}, t), \quad (35)$$

$$\begin{aligned} \mathbf{E}_{\text{ind}}(A_{\text{eff}} - a; \mathbf{r}, t) &= \frac{4\pi\hbar}{e^2} \hat{z} \times \delta\mathbf{j}_a(A_{\text{eff}} - a; \mathbf{r}, t) + e \nabla \int d\mathbf{r}' \\ &\quad \times v(\mathbf{r} - \mathbf{r}') \delta\rho_{ac}(A_{\text{eff}} - a; \mathbf{r}', t), \end{aligned} \quad (36)$$

where

$$\begin{aligned} \delta\rho_{ac}(A_{\text{eff}} - a; \mathbf{r}, t) &= \rho_{ac}(A_{\text{eff}} + A_{\text{tot}} - a; \mathbf{r}, t) \\ &\quad \times \exp \left(i \frac{e}{\hbar} \int_{\mathbf{r}'}^{\mathbf{r}} d\mathbf{r}'' \cdot \mathbf{A}_{\text{tot}} \right) \\ &\quad - \rho_{ac}^{\text{eq}}(A_{\text{eff}} - a; \mathbf{r}, t), \end{aligned} \quad (37)$$

$$\delta\mathbf{j}_a(A_{\text{eff}} - a; \mathbf{r}, t) = \mathbf{j}_a(A_{\text{eff}} + A_{\text{tot}} - a; \mathbf{r}, t) - \mathbf{j}_a^{\text{eq}}(A_{\text{eff}} - a; \mathbf{r}, t) \quad (38)$$

are the changes in the density and the current, respectively, induced by the total field A_{tot} . The phase factor $\exp[i(e/\hbar) \int_{\mathbf{r}'}^{\mathbf{r}} d\mathbf{r}'' \cdot \mathbf{A}_{\text{tot}}]$ has been introduced in order to let $\delta\rho_{ac}$ refer to the equilibrium state. For instance $\delta\mathbf{j}_a$ results when $\delta\rho_{ac}$ is substituted for ρ_{ac}^{eq} in the expression for the

equilibrium current, Eq. (19). Equations (35) and (36) are what are expected on physical grounds.¹⁷ There is a magnetic field associated with a change in the density of composite fermions and thus the amount of attached flux. There is also a Faraday electric field associated with a current of attached flux. The last term in Eq. (36) is due to the electron-electron interaction.

Golubev and Zaikin¹ did not perform this change of variables, since the difference between the external and the total field is not of great bearing in their application of the formalism to weak localization. The second term in their formula (46), which they neglect, is exactly what is covered by this change of variable. For composite fermions such a neglect is not permissible. For instance, the Hall conductance¹⁵ due to the Faraday electric field induced by the current—the first term in Eq. (36). The mechanism is completely analogous to the treatment in the case of the Boltzmann equation.¹⁸

The Jacobian associated with the variable change from a to $a - A_{\text{ind}}(a)$ in Eqs. (16) and (17) is the identity 1, since both the density and the current, and thus $A_{\text{ind}}(a)$, are retarded functions of the gauge field $A_{\text{eff}} - a$. Consequently the change in the physical density matrix

$$\begin{aligned} \delta\rho(\mathbf{r}, \mathbf{r}'; t) = & \left\langle \delta\rho_{ac}(A_{\text{eff}} - a, \mathbf{r}, \mathbf{r}'; t) \right. \\ & \left. \times \exp\left\{ i \frac{e}{\hbar} \int_{\mathbf{r}'}^{\mathbf{r}} d\mathbf{r}'' \cdot [\mathbf{A}_{\text{eff}}(\mathbf{r}'') - \bar{\mathbf{a}}(\mathbf{r}'', t)] \right\} \right\rangle_{a,t}^{\text{eq}}. \end{aligned} \quad (39)$$

Notice that so far no linearization in the applied field has been performed and everything is exact.

The average $\langle X \rangle_{a,t}^{\text{eq}}$, Eq. (17), we know how to calculate, Eqs. (25) and (26). The linear response of ρ_{ac} to the total field A_{tot} , $\delta\rho_{ac}(A_{\text{eff}} - a; \mathbf{r}, \mathbf{r}', t)$, is obtained in the next subsection by solving Eq. (31) linearized in A_{tot} :

$$\begin{aligned} i\hbar \partial_t \delta\rho_{ac}(\mathbf{r}, \mathbf{r}'; t) \approx & \int d\mathbf{r}'' H^+(\mathbf{r}, \mathbf{r}'', t) \delta\rho_{ac}(\mathbf{r}'', \mathbf{r}'; t) \\ & - \int d\mathbf{r}'' \delta\rho_{ac}(\mathbf{r}, \mathbf{r}'', t) H^-(\mathbf{r}'', \mathbf{r}', t) \\ & + i\hbar D(\mathbf{r}, \mathbf{r}', t), \end{aligned} \quad (40)$$

where the explicit dependence on the gauge fields $A_{\text{eff}} - a$ and A_{tot} is suppressed in the notation. New one-particle Hamiltonians

$$w_a(A_{\text{eff}} - a; \mathbf{R}, \mathbf{p}, t) = \int d\Delta\mathbf{r} \exp\left(-\frac{i}{\hbar} \int_{\mathbf{R}-\Delta\mathbf{r}/2}^{\mathbf{R}+\Delta\mathbf{r}/2} d\mathbf{r}'' \cdot [\mathbf{p} - e\mathbf{A}_{\text{eff}}(\mathbf{r}'') + e\bar{\mathbf{a}}(\mathbf{r}'', t)] \right) \rho_{ac}\left(A_{\text{eff}} - a; \mathbf{R} + \frac{1}{2}\Delta\mathbf{r}, \mathbf{R} - \frac{1}{2}\Delta\mathbf{r}; t \right). \quad (45)$$

In the semiclassical limit it corresponds to a field-dependent phase-space distribution function. We will not consider effects beyond the approximation where it is the Fermi distribution function.

F. Single-particle path integral in linear response

In this subsection we obtain a formal representation of $\delta\rho$ in the linear response in terms of a single-particle path integrals. The first step is to notice that the solution to Eq. (40) can be written in terms of evolution operators U^μ :

$$H^+ = \frac{H(+)+H(-)}{2} \Big|_{\phi=0} + \left(\frac{1}{2} - \rho_{ac} \right) [H(+)-H(-)] \Big|_{\phi=0}, \quad (41)$$

$$H^- = \frac{H(+)+H(-)}{2} \Big|_{\phi=0} - [H(+)-H(-)] \left(\frac{1}{2} - \rho_{ac} \right) \Big|_{\phi=0} \quad (42)$$

have been introduced in analogy with the equation of motion for the physical one-particle density matrix $i\hbar \partial_t \rho = [H, \rho]$. $H^+ \neq H^-$ because the Chern-Simons field is different on the forward and backward contours. The inhomogeneous stimulus $D = D_1 + D_2$ is split into two parts, each of which are gauge invariant in the total applied field A_{tot} . D_2 contains all the dependence upon the difference gauge field $\Delta\mathbf{a}$, while D_1 is independent of $\Delta\mathbf{a}$. In Appendix B the explicit expressions for D_1 and D_2 are listed. In linear response D_1 and D_2 contribute additively to the physical density matrix. The response due to D_2 we will neglect, since it vanishes in the approximation where D is replaced by its average value $\langle D \rangle_{a,t}^{\text{eq}}$. Below we do not go beyond this approximation.

In the real-space representation used above D is not gauge invariant in $A_{\text{eff}}(\mathbf{r}) - \bar{\mathbf{a}}$, so we find it convenient to change representation to the gauge-invariant Wigner function set of variables. In analogy with Eqs. (4), (5), and (7),

$$\begin{aligned} D(\mathbf{r}, \mathbf{r}', t) = & \int \frac{d\mathbf{p}}{(2\pi\hbar)^2} \\ & \times \exp\left(\frac{i}{\hbar} \int_{\mathbf{r}'}^{\mathbf{r}} d\mathbf{r}'' \cdot [\mathbf{p} - e\mathbf{A}_{\text{eff}}(\mathbf{r}'') + e\bar{\mathbf{a}}(\mathbf{r}'', t)] \right) \\ & \times D\left[\frac{1}{2}(\mathbf{r} + \mathbf{r}'), \mathbf{p} \right]. \end{aligned} \quad (43)$$

When the generalized density matrix $\rho_{ac}(\mathbf{r}, \mathbf{r}')$ is peaked around $\mathbf{r} = \mathbf{r}'$ on a length scale which is much smaller than the typical scale of variation of the total driving field, the gauge invariant D_1 and thus D are approximately equal to the standard Boltzmann equation driving term

$$D(\mathbf{R}, \mathbf{p}) \approx -e \left(\frac{1}{m} \mathbf{p} \times \hat{\mathbf{z}} B_{\text{tot}} + \mathbf{E}_{\text{tot}} \right) \cdot \frac{\partial}{\partial \mathbf{p}} w_a(A_{\text{eff}} - a; \mathbf{R}, \mathbf{p}). \quad (44)$$

In the presence of the gauge field $A_{\text{eff}} - a$ the Wigner distribution function $w_a(A_{\text{eff}} - a; \mathbf{R}, \mathbf{p}, t)$ is defined in analogy with Eqs. (5) and (7):

$$\delta\rho_{ac}(\mathbf{r},\mathbf{r}';t)=-\frac{i}{\hbar}\int_{-\infty}^t dt_1 \int d\mathbf{r}_1 \int d\mathbf{r}_2 U^+(\mathbf{r},t,\mathbf{r}_1,t_1)D(\mathbf{r}_1,\mathbf{r}_2,t_1)U^-(\mathbf{r}_2,t,\mathbf{r}',t). \quad (46)$$

The evolution operators

$$U^\mu(t,t')=\hat{T}^\mu\exp\left[-\frac{i}{\hbar}\int_{t'}^t dt'' H^\mu(t'')\right], \quad \mu=\pm, \quad (47)$$

where \hat{T}^+ and \hat{T}^- are the time and antitime ordering operators, respectively. U^+ and U^- can be represented as usual single-particle phase-space path integrals

$$U^\mu(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)=\int_{\mathbf{r}(t_2)=\mathbf{r}_2}^{\mathbf{r}(t_1)=\mathbf{r}_1} \mathcal{D}\mathbf{r} \int \mathcal{D}\mathbf{p} \exp\left\{\frac{i}{\hbar}\int_{t_2}^{t_1} dt[(\mathbf{p}-eA_{\text{eff}}+e\bar{\mathbf{a}})\cdot\dot{\mathbf{r}}-H^\mu(\mathbf{r}(s),\mathbf{p}(s),s)]\right\}, \quad \mu=\pm. \quad (48)$$

The response of the physical Wigner function is then [Eqs. (7), (39), and (46)]

$$\delta w(\mathbf{R},\mathbf{p},t)=\int_{-\infty}^t dt_1 \int d\mathbf{R}_1 \int \frac{d\mathbf{p}_1}{(2\pi\hbar)^2} \langle J(\mathbf{R},\mathbf{p},t;\mathbf{R}_1,\mathbf{p}_1,t_1)D(\mathbf{R}_1,\mathbf{p}_1,t_1) \rangle_{a,t}^{\text{eq}}, \quad (49)$$

where

$$\begin{aligned} J(\mathbf{R},\mathbf{p},t;\mathbf{R}_1,\mathbf{p}_1,t_1) &= \int d\Delta\mathbf{r} \int d\Delta\mathbf{r}_1 \int d\mathbf{r}'_1 \exp\left(-\frac{i}{\hbar}\int_{\mathbf{R}+\Delta\mathbf{r}/2}^{\mathbf{R}-\Delta\mathbf{r}/2} d\mathbf{r}'' \cdot [\mathbf{p}-eA_{\text{eff}}(\mathbf{r}'') + e\bar{\mathbf{a}}(\mathbf{r}'',t)]\right) \\ &\quad \times \exp\left(\frac{i}{\hbar}\int_{\mathbf{R}_1+\Delta\mathbf{r}_1/2}^{\mathbf{R}_1-\Delta\mathbf{r}_1/2} d\mathbf{r}'' \cdot [\mathbf{p}_1-eA_{\text{eff}}(\mathbf{r}'') + e\bar{\mathbf{a}}(\mathbf{r}'',t_1)]\right) \\ &\quad \times U^+\left(\mathbf{R}+\frac{\Delta\mathbf{r}}{2},t;\mathbf{R}_1+\frac{\Delta\mathbf{r}_1}{2},t_1\right)U^-\left(\mathbf{R}_1-\frac{\Delta\mathbf{r}_1}{2},t_1;\mathbf{R}-\frac{\Delta\mathbf{r}}{2},t\right) \end{aligned} \quad (50)$$

is the propagator of the stimulus D in the Wigner representation.

Inserting Eq. (48) we finally arrive at the following contour path integral representation for the propagator J :

$$\begin{aligned} J(\mathbf{R},\mathbf{p},t;\mathbf{R}_1,\mathbf{p}_1,t_1) &= \int d\Delta\mathbf{r} \int d\Delta\mathbf{r}_1 \int d\mathbf{r}'_1 \int \mathcal{D}\mathbf{r}(+) \\ &\quad \times \int \mathcal{D}\mathbf{p}(+) \int \mathcal{D}\mathbf{r}(-) \int \mathcal{D}\mathbf{p}(-) \\ &\quad \times \exp\left(\frac{i}{\hbar}S_0 + \frac{i}{\hbar}\Delta S\right), \end{aligned} \quad (51)$$

where

$$\begin{aligned} S_0 &= \oint d\sigma \dot{\mathbf{r}} \cdot \{\mathbf{p}(\sigma) - eA_{\text{eff}}[\mathbf{r}(\sigma),s]\} \\ &\quad - \int_{t_1}^t ds H_0[\mathbf{r}(s,+),\mathbf{p}(s,+),s] \\ &\quad + \int_{t_1}^t ds H_0[\mathbf{r}(s,-),\mathbf{p}(s,-),s] \end{aligned} \quad (52)$$

is the action for noninteracting particles in an external gauge field A_{eff} . The closed contour is depicted in Fig. 2. All the contributions related to the Chern-Simons gauge field are gathered in the second term in the exponent of Eq. (51):

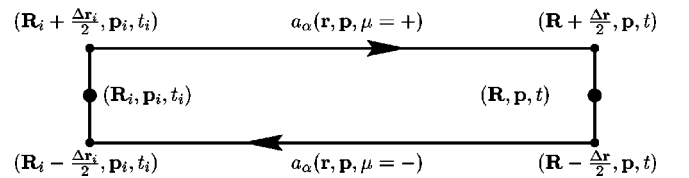


FIG. 2. The closed contour path $(\mathbf{r}(\sigma),\mathbf{p}(\sigma))$ starts out at $\sigma=(t_1,+)$ at $(\mathbf{R}_1+\Delta\mathbf{r}_1/2,\mathbf{p}_1)$ and runs along $(\mathbf{r}(s,+),\mathbf{p}(s,+))$ up to $(\mathbf{R}+\Delta\mathbf{r}/2,\mathbf{p})$ at contour time $\sigma=(t,+)$. Here it runs along the straight line path from $\mathbf{R}+\Delta\mathbf{r}/2$ to $\mathbf{R}-\Delta\mathbf{r}/2$ with the constant momentum \mathbf{p} and at constant σ . From here it then runs back along $(\mathbf{r}(s,-),\mathbf{p}(s,-))$ to $(\mathbf{R}_1-\Delta\mathbf{r}_1/2,\mathbf{p}_1)$ at contour time $\sigma=(t_1,-)$. Finally it runs along a straight line path back to $\mathbf{R}_1+\Delta\mathbf{r}_1/2$ with the constant momentum \mathbf{p}_1 and at a constant time t_1 .

$$\begin{aligned}
\Delta S = & + e \oint \{ d\mathbf{r} \cdot \bar{a}[\mathbf{r}(\sigma), s] - d\sigma \bar{a}_0[\mathbf{r}(\sigma), s] \} + e \int_{t_i}^t ds \left\{ \frac{1}{2} - w_a[\mathbf{r}(s, +), \mathbf{p}(s^+, +), s] \right\} \\
& \times \left\{ \frac{\mathbf{p}(s, +)}{m} \cdot \Delta \mathbf{a}[\mathbf{r}(s, +), s] - \Delta a_0[\mathbf{r}(s, +), s] \right\} + e \int_{t_i}^t ds \left\{ \frac{1}{2} - w_a[\mathbf{r}(s, -), \mathbf{p}(s^+, -), s] \right\} \\
& \times \left\{ \frac{\mathbf{p}(s, -)}{m} \cdot \Delta \mathbf{a}[\mathbf{r}(s, -), s] - \Delta a_0[\mathbf{r}(s, -), s] \right\} - \frac{e^2}{8m} \int_{t_i}^t ds \{ \Delta \mathbf{a}[\mathbf{r}(s, +), s]^2 - \Delta \mathbf{a}[\mathbf{r}(s, -), s]^2 \} \\
& + \frac{e}{2m} \frac{\hbar}{i} \int_{t_i}^t ds \{ \Delta \mathbf{a}[\mathbf{r}(s, +), s] \cdot \nabla w_a[\mathbf{r}(s, +), \mathbf{p}(s^+, +), s] + \Delta \mathbf{a}[\mathbf{r}(s, -), s] \cdot \nabla w_a[\mathbf{r}(s, -), \mathbf{p}(s^+, -), s] \}. \quad (53)
\end{aligned}$$

Here s^+ on both the forward and backward in time contours is a bit closer to the observation time t than s . The only place where this small time shift is important is in $w_a(\mathbf{r}, \mathbf{p}(s^+), s)$, the Wigner distribution function corresponding to ρ_{ac} , which is likely to be a rapidly varying function of the momentum \mathbf{p} at the Fermi surface. Note that \mathbf{p} in the semiclassical limit is the kinetic momentum in the gauge field $A_{\text{eff}} - \bar{a}$.

In the rest of this article we are only going to consider the approximation obtained by factorizing Eq. (49):

$$\delta w(\mathbf{R}, \mathbf{p}, t) = \int_{-\infty}^t dt_i \int d\mathbf{R}_i \int \frac{d\mathbf{p}_i}{(2\pi\hbar)^2} \langle J(\mathbf{R}, \mathbf{p}, t; \mathbf{R}_i, \mathbf{p}_i, t_i) \rangle_{a,t}^{\text{eq}} \langle D(\mathbf{R}_i, \mathbf{p}_i, t_i) \rangle_{a,t}^{\text{eq}}. \quad (54)$$

Above we argued that the stimulus $\langle D \rangle_{a,t}^{\text{eq}}$ in the semiclassical limit is the standard Boltzmann driving term, Eq. (44). We calculate the average of the propagator J perturbatively to second order in the Chern-Simons gauge field a . This procedure is gauge invariant. First notice that $\langle \Delta S \rangle_{a,t}^{\text{eq}} = 0$, since $\langle \bar{a} = 0 \rangle_{a,t}^{\text{eq}} = 0$ and all averages with a Δa at the latest time vanishes. To second order in the Chern-Simons field fluctuations is

$$\langle J(\mathbf{R}, \mathbf{p}, t; \mathbf{R}_i, \mathbf{p}_i, t_i) \rangle_{a,t}^{\text{eq}} = \int d\Delta \mathbf{r} \int d\Delta \mathbf{r}_i \int d\mathbf{r}'_i \int \mathcal{D}\mathbf{r}(+) \int \mathcal{D}\mathbf{p}(+) \int \mathcal{D}\mathbf{r}(-) \int \mathcal{D}\mathbf{p}(-) \exp\left(\frac{i}{\hbar} S_0 + \frac{i}{\hbar} S_{\text{int}}\right), \quad (55)$$

where

$$S_{\text{int}}(t_i \rightarrow t) = \frac{i}{2\hbar} \langle \Delta S^2 \rangle_{a,t}^{\text{eq}} = \sum_{\mu', \mu'' = \pm} \mu' \mu'' \int_{t_i}^t ds' \int_{t_i}^{s'} ds'' L_{\text{int}}(s', \mu'; s'', \mu''). \quad (56)$$

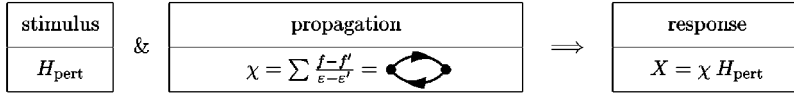
For later use we have incorporated the time interval (t_i, t) in which the interaction takes place into the notation for the action and introduced the retarded Lagrangian $L_{\text{int}} = L_{\text{int}}(s', \mu'; s'', \mu'')$:

$$\begin{aligned}
L_{\text{int}} = & i\Theta(s' - s'')(-1, \dot{\mathbf{r}}(s', \mu')) \bar{\mathbf{C}}(\mathbf{r}(s', \mu'), s'; \mathbf{r}(s'', \mu''), s'') \begin{pmatrix} -1 \\ \dot{\mathbf{r}}(s'', \mu'') \end{pmatrix} \\
& - \mu'' \left\{ \frac{1}{2} - w[\mathbf{p}(s''^+, \mu'')] \right\} (-1, \dot{\mathbf{r}}(s', \mu')) \bar{\mathbf{D}}(\mathbf{r}(s', \mu'), s'; \mathbf{r}(s'', \mu''), s'') \begin{pmatrix} -1 \\ \frac{1}{m} \mathbf{p}(s'', \mu'') \end{pmatrix}. \quad (57)
\end{aligned}$$

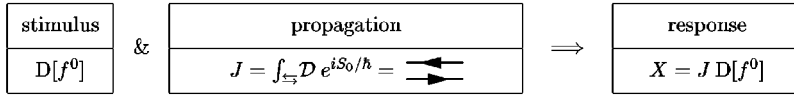
We have used that to lowest order $w_a(a; \mathbf{R}, \mathbf{p}, t) = w_a(a; \mathbf{R}, \mathbf{p}, t)|_{a=0} = w(\mathbf{p}) = f_{\text{F}}(\epsilon_{\mathbf{p}})$, where $f_{\text{F}}(\epsilon)$ is the Fermi distribution function. The 3×3 matrices $\bar{\mathbf{C}}$ and $\bar{\mathbf{D}}$ are defined in Eqs. (25) and (26). In the Coulomb gauge their most singular part is the transverse part. Below we only discuss this most singular part, i.e., the transverse contribution to L_{int} ,

$$\begin{aligned}
\frac{i}{\hbar} L_{\text{int}\perp\perp} = & \Theta(s' - s'') \frac{1}{\hbar} \int \frac{d\mathbf{q}}{(2\pi\hbar)^2} \int \frac{d\omega}{2\pi} e^{-i\omega(s' - s'')} \exp\left[\frac{i}{\hbar} \mathbf{q} \cdot [\mathbf{r}(s', \mu') - \mathbf{r}(s'', \mu'')]\right] \\
& \times \frac{1}{q^2} [\mathbf{q} \times \dot{\mathbf{r}}(s', \mu')] \cdot [\mathbf{q} \times \dot{\mathbf{r}}(s'', \mu'')] \coth\left(\frac{\hbar\omega}{2k_{\text{B}}T}\right) \text{Im} D_{\perp\perp}(\mathbf{q}, \omega) - \mu'' \frac{i}{\hbar} \Theta(s' - s'') \\
& \times \int \frac{d\mathbf{q}}{(2\pi\hbar)^2} \int \frac{d\omega}{2\pi} \exp[-i\omega(s' - s'')] \exp\left[\frac{i}{\hbar} \mathbf{q} \cdot [\mathbf{r}(s', \mu') - \mathbf{r}(s'', \mu'')]\right] \\
& \times \frac{1}{mq^2} [\mathbf{q} \times \dot{\mathbf{r}}(s', \mu')] \cdot [\mathbf{q} \times \mathbf{p}(s'', \mu'')] \frac{1}{2} D_{\perp\perp}(\mathbf{q}, \omega) \{1 - 2w[\mathbf{p}(s''^+, \mu'')]\}. \quad (58)
\end{aligned}$$

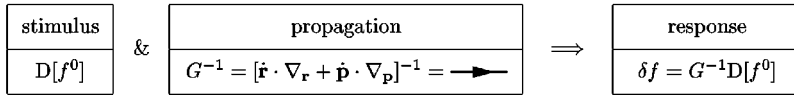
The Kubo formula



The path integral formalism



The Boltzmann equation



In the second term $\frac{1}{2}D_{\perp\perp}$ can be replaced by either $\text{Re}D_{\perp\perp}$ or $\text{Im}D_{\perp\perp}$ as long as we remember that L_{int} is retarded ($s' > s''$).

G. Recapitulation of the major results

Summing up, the linear response of the one-particle density matrix to an external perturbation is given by Eq. (49), which schematically can be written as

$$\delta w = \int \langle JD \rangle_{a,t}^{\text{eq}}. \quad (59)$$

This expression we approximately factorize in order to arrive at Eq. (54) or, schematically,

$$\delta w = \int \langle J \rangle_{a,t}^{\text{eq}} \langle D \rangle_{a,t}^{\text{eq}}. \quad (60)$$

In the semiclassical limit the stimulus $\langle D \rangle_{a,t}^{\text{eq}}$ is in the classical limit given by the driving term in the Boltzmann equation (44). In Eq. (55) the propagator $\langle J \rangle_{a,t}^{\text{eq}}$ is represented as a double single-particle path integral with an action similar to the familiar Caldeira-Leggett influence functional. Schematically we write

$$\langle J \rangle_{a,t}^{\text{eq}} = \int \mathcal{D}(\text{path}+) \int \mathcal{D}(\text{path}-) \exp\left[\frac{i}{\hbar}(S_0 + S_{\text{int}})\right]. \quad (61)$$

Here S_0 is the usual action for noninteracting particles, Eq. (52). It does not couple the forward path $+$ and the backward path $-$. S_{int} contains the contribution from the gauge fields and the interactions, and it couples the two paths. It is given by Eqs. (56) and (57). The imaginary part of S_{int} is governed by the fluctuations, Eq. (26), while the real part is determined by the retarded gauge field propagator, Eq. (25). The distribution function also enters into the real part.

III. COMPARISON WITH THE KUBO FORMULA

In equilibrium a standard way to perform linear response is to use the Kubo formula. Though the path integral formalism and the Kubo formula on a formal level are similar, there are some important differences in the descriptions. This is most clearly illustrated by considering a noninteracting sys-

tem as is done in Fig. 3, where we also compare to the classical Boltzmann equation. The linear response is in all three cases split into a stimulus that is propagated in order to give the response. In the Kubo formula the stimulus is the applied potential, while in the path integral formalism it is the induced excitation, which in the semiclassical limit equals the usual Boltzmann stimulus, Eq. (44). Despite that, the stimulus is very different the Kubo formula and the path integral formulations have similar pictorial representations of the propagation: in both cases there is a forward and backward propagating electron line. However, in the Kubo formula a line mathematically represents a Green's function and the propagator in the noninteracting case is the Lindhard function,¹⁹ while in the noninteracting path integral formalism a line is a conventional free particle Feynman path integral. In the semiclassical limit the double Feynman path integral reduces to the classical path integral propagator for the Boltzmann equation.²⁰ The difference between the Kubo formula and the path integral formalism is highlighted by the observation that the (Fermi) distribution function in the Kubo formula is contained entirely within the propagator, while in the path integral formalism it is the stimulus and not the propagation that depends on the distribution function for a noninteracting system. Notice also that in the path integral formulation no statements are made about equilibrium. The fact that for a noninteracting system the propagation is independent of the actual distribution function is well known from the Landauer-Büttiker scattering approach to transport through a noninteracting part of the system.

Within the Kubo formalism it is very natural to treat the interactions perturbatively. The effect of the interaction is traditionally divided into self-energy contributions, which alters the propagation properties of a single line or Green's function, and vertex corrections that connect the two lines. It is possible to make the same distinction in a perturbative treatment of the interactions in the path integral formulation, and we will also use the terms self-energy and vertex corrections to distinguish between interactions that only involve one path and those that connect the forward and backward propagating paths. For an interacting system the propagator in the path integral formalism thus also depends on the actual distribution function. This is well known in the classical limit of the Boltzmann equation, since the propagator then depends on both the collision integral and the Landau interac-

tion function, but still the stimulus has already gathered the most trivial dependence upon the distribution function.

IV. SEMICLASSICAL LIMIT AND THE SEPARATION BETWEEN THE TWO PATHS

Traditionally the semiclassical limit of a double path integral is the limit where the important contributions are those where the forward and backward paths are almost identical. In a discussion of the semiclassical limit it is thus important to know how far apart on average the forward and backward paths are. This information is contained in the moments of the difference variable $\Delta\mathbf{r}(s)$. For a noninteracting system or more generally a system with instantaneous interactions the standard deviation of $\Delta\mathbf{r}$ can be estimated from the Heisenberg uncertainty relations between $\Delta\mathbf{r}$ and its conjugate variable, the average momentum \mathbf{P} . That \mathbf{P} indeed is the conjugate variable can be read off from the relations

$$\frac{\partial}{\partial \mathbf{P}_f} S_0(\mathbf{R}_f, \mathbf{P}_f, t; \mathbf{R}_i, \mathbf{P}_i, 0) = -\Delta\mathbf{r}(t), \quad (62)$$

$$\frac{\partial}{\partial \mathbf{P}_i} S_0(\mathbf{R}_f, \mathbf{P}_f, t; \mathbf{R}_i, \mathbf{P}_i, 0) = \Delta\mathbf{r}(0), \quad (63)$$

where S_0 is the action for a noninteracting system, Eq. (52). We are mainly interested in Fermi liquids, so we prefer to use a polar coordinate system representation of the average momenta:

$$\mathbf{P}(s) = P(s) \begin{pmatrix} \cos \phi(s) \\ \sin \phi(s) \end{pmatrix}, \quad (64)$$

where the angle along the Fermi surface ϕ is measured relative to an arbitrarily fixed x axis. The conjugate variable to the size of the momentum is

$$\frac{\partial}{\partial P_f} S_0(\mathbf{R}_f, \mathbf{P}_f, t; \mathbf{R}_i, \mathbf{P}_i, 0) = -\frac{1}{P_f} \mathbf{P}_f \cdot \Delta\mathbf{r}(t). \quad (65)$$

From the Heisenberg uncertainty relation we then expect the standard deviation of $(1/P)\mathbf{P} \cdot \Delta\mathbf{r}$, the difference along the path, to be given by \hbar divided by the uncertainty in the size of the momentum, i.e., $(1/P)\mathbf{P} \cdot \Delta\mathbf{r} \approx \hbar v_F (k_B T)^{-1}$. This argument is carried through in more detail in Appendix C and the coefficient to $\hbar v_F (k_B T)^{-1}$ is also shown to be $1/\sqrt{3}$.

The conjugate variable to the angle ϕ is

$$\frac{\partial}{\partial \phi_f} S_0(\mathbf{R}_f, \mathbf{P}_f, t; \mathbf{R}_i, \mathbf{P}_i, 0) = -\mathbf{P}_f \times \Delta\mathbf{r}(t). \quad (66)$$

Remembering that the relevant momentum P_f approximately is the Fermi momentum we will denote the quantity $p_F^{-1} \mathbf{P}_f \times \Delta\mathbf{r}$ by the width. The Heisenberg uncertainty relation then says that the standard deviation of the width is approximately the Fermi wavelength λ_F divided by the angular uncertainty in the propagation direction. For a well-collimated beam the width is thus big, while it is small (of the order of λ_F) when measuring the current response to a homogeneous field, since in this case the angular uncertainty is as big as possible (of order 1).

The separation between the paths is not in itself directly measurable, but it tends to show up indirectly in experiments probing the semiclassical regime. It marks, e.g., the cross-over between the long-wavelength regime, where the vertex corrections are comparable to and partially cancel the self-energy, and the short-wavelength regime, where the vertex corrections play a minor role. For instance in the calculation of the inverse transport time due to impurity scattering in the famous factor $(1 - \cos \theta)$ the scattering angle θ can be interpreted as $(1/\hbar)q\lambda_F$, where q is the scattering vector and, as we saw above, λ_F is the width. For the composite fermions, where the force is transmitted via a magnetic field and thus a flux through an area, we show in Sec. V that the width acts as an effective cutoff in the conventional calculations of the effective mass.

The above considerations regarding the width are only true for a system with a time-local action. For composite fermions this is not the case, since the important transverse fluctuations have a characteristic frequency around $g_1 q^2$; see Eq. (29). We thus have to take the lowest-order correction due to the terms in the action that are not time local into account. We do this perturbatively.

To this end let us define the average width squared $(\hbar/q_\perp^c)^2$ at time s as

$$\left(\frac{\hbar}{q_\perp^c} \right)^2 \equiv \langle |p_F^{-2} [\mathbf{P}(s) \times \Delta\mathbf{r}(s)]^2| \rangle_{[0,t]}, \quad (67)$$

where the average $\langle |X| \rangle_{[0,t]}$ is defined as

$$\langle |X| \rangle_{[0,t]} = \frac{\int \frac{d\mathbf{P}_f}{(2\pi\hbar)^2} \int d\mathbf{R}_f \int \frac{d\mathbf{P}_i}{(2\pi\hbar)^2} \int d\mathbf{R}_i \mathcal{R}(\mathbf{P}_f, \mathbf{R}_f) \mathcal{D}(\mathbf{P}_i, \mathbf{R}_i) \left\langle \left\langle X \exp \left[\frac{i}{\hbar} S_0(\mathbf{R}_f, \mathbf{P}_f, t; \mathbf{R}_i, \mathbf{P}_i, 0) \right] \right\rangle \right\rangle_{[0,t]}}{\int \frac{d\mathbf{P}_f}{(2\pi\hbar)^2} \int d\mathbf{R}_f \int \frac{d\mathbf{P}_i}{(2\pi\hbar)^2} \int d\mathbf{R}_i \mathcal{R}(\mathbf{P}_f, \mathbf{R}_f) \mathcal{D}(\mathbf{P}_i, \mathbf{R}_i) \left\langle \left\langle \exp \left[\frac{i}{\hbar} S_0(\mathbf{R}_f, \mathbf{P}_f, t; \mathbf{R}_i, \mathbf{P}_i, 0) \right] \right\rangle \right\rangle_{[0,t]}}. \quad (68)$$

Here $\langle \langle X \rangle \rangle_{[0,t]}$ is a shorthand notation for the path integral over paths starting at time 0 and ending at time t with the Wigner function variables as boundary conditions. $\langle \langle X \rangle \rangle_{[0,t]}$ is defined such as to include the action $S_{\text{int}}(t, 0)$ arising from the interactions within the interval $[0, t]$:

$$\langle \langle X \rangle \rangle_{[0,t]} = \int_{[0,t]} \mathcal{D}\mathbf{R} \int_{[0,t]} \mathcal{D}\mathbf{P} \int_{[0,t]} \mathcal{D}\Delta\mathbf{p} \int_{[0,t]} \mathcal{D}\Delta\mathbf{r} \exp \left(\frac{i}{\hbar} S_{\text{int}}(t, 0) \right) X. \quad (69)$$

In the semiclassical limit the stimulus $D(\mathbf{P}_i, \mathbf{R}_i)$ is given by Eq. (44). For a current measurement the response $\mathcal{R}(\mathbf{P}_f, \mathbf{R}_f) = -e\mathbf{v}_f\Omega^{-1} = -e(1/m)\mathbf{p}_f\Omega^{-1}$. Ω is the volume of the sample. In the long-time limit the cutoff turns out to be independent of t and s and t can be taken to be infinity or rather a scattering time. When no confusion is possible we will omit the index $[0, t]$. Notice that according to Eq. (55) the last factor in the denominator is

$$\left\langle \left\langle \exp \left[\frac{i}{\hbar} S_0(t, \mathbf{R}_f, \mathbf{P}_f, t; \mathbf{R}_i, \mathbf{P}_i, 0) \right] \right\rangle \right\rangle_{[0, t]} = \langle J(\mathbf{R}_f, \mathbf{P}_f, t; \mathbf{R}_i, \mathbf{P}_i, 0) \rangle_{a, t}. \quad (70)$$

It could easily be argued that it is better to use the absolute value of both the response $\mathcal{R}(\mathbf{P}_f, \mathbf{R}_f)$ and the stimulus $D(\mathbf{P}_i, \mathbf{R}_i)$ in the definition of $\langle |X| \rangle$, Eq. (68), since the response might be the sum of a positive contribution from paths originating in one part of the stimulus phase-space and a negative contribution from other parts of the phase-space. However, the distinction does not seem to be terribly important in the examples discussed below, so for simplicity we stick with the above definition.

Ultimately we perform a perturbative expansion in the scattering events that begin before the observation time s and finish later than s . We thus split the action due to the interaction S_{int} , Eq. (56), in the average $\langle |X| \rangle_{[0, t]}$ into three parts

$$S_{\text{int}}(t, 0) = S_{\text{int}}(s, 0) + S_{\text{int}}(t, s) + \sum_{\mu', \mu'' = \pm} \mu' \mu'' \int_s^t ds' \int_0^s ds'' L_{\text{int}}, \quad (71)$$

where the Lagrangian $L_{\text{int}} = L_{\text{int}}(s', \mu'; s'', \mu'')$ only considering the transverse contribution given by Eq. (58). Also using the conjugate variable relationship, Eq. (66), to rewrite the width squared, Eq. (67), we finally obtain

$$\begin{aligned} \left(\frac{\hbar}{q_{\perp}^c} \right)^2 &= \lambda_{\text{F}}^2 \mathcal{N} \int \frac{d\mathbf{P}_f}{(2\pi\hbar)^2} \int d\mathbf{R}_f \int \frac{d\mathbf{P}_i}{(2\pi\hbar)^2} \int d\mathbf{R}_i \mathcal{R}(\mathbf{P}_f, \mathbf{R}_f) D(\mathbf{P}_i, \mathbf{R}_i) \int \frac{d\mathbf{P}(s)}{(2\pi\hbar)^2} \int d\mathbf{R}(s) \\ &\times \left\langle \left\langle \left\langle \left\langle \exp \left(\frac{i}{\hbar} \sum_{\mu', \mu'' = \pm} \mu' \mu'' \int_s^t ds' \int_0^s ds'' L_{\text{int}} \right) \frac{\partial}{\partial \phi(s)} \right. \right. \right. \\ &\times \exp \left(\frac{i}{\hbar} S_0[\mathbf{R}(s), \mathbf{P}(s), s; \mathbf{R}_i, \mathbf{P}_i, 0] \right) \frac{\partial}{\partial \phi(s)} \exp \left(\frac{i}{\hbar} S_0[\mathbf{R}_f, \mathbf{P}_f, t; \mathbf{R}(s), \mathbf{P}(s), s] \right) \left. \left. \right\rangle \right\rangle_{[0, s]} \left. \right\rangle_{[s, t]}, \quad (72) \end{aligned}$$

where the normalization is contained inside the new constant \mathcal{N} :

$$\mathcal{N}^{-1} = \int \frac{d\mathbf{P}_f}{(2\pi\hbar)^2} \int d\mathbf{R}_f \int \frac{d\mathbf{P}_i}{(2\pi\hbar)^2} \int d\mathbf{R}_i \mathcal{R}(\mathbf{P}_f, \mathbf{R}_f) D(\mathbf{P}_i, \mathbf{R}_i) \left\langle \left\langle \exp \left[\frac{i}{\hbar} S_0(\mathbf{R}_f, \mathbf{P}_f, t; \mathbf{R}_i, \mathbf{P}_i, 0) \right] \right\rangle \right\rangle_{[0, t]}. \quad (73)$$

In the next two subsections we perturbatively calculate the lowest-order contribution from the non-time-local term $\sum_{\mu', \mu'' = \pm} \mu' \mu'' \int_s^t ds' \int_0^s ds'' L_{\text{int}}$ to the width, Eq. (72).

A. Zeroth order or the time-local limit revisited

To lowest order or in the so-called time-local approximation, the last term in Eq. (71) vanishes and the width squared is

$$\begin{aligned} \left(\frac{\hbar}{q_{\perp}^c} \right)^2 &= \lambda_{\text{F}}^2 \mathcal{N} \int \frac{d\mathbf{P}_f}{(2\pi\hbar)^2} \int d\mathbf{R}_f \int \frac{d\mathbf{P}_i}{(2\pi\hbar)^2} \int d\mathbf{R}_i \mathcal{R}(\mathbf{P}_f, \mathbf{R}_f) D(\mathbf{P}_i, \mathbf{R}_i) \int \frac{d\mathbf{P}(s)}{(2\pi\hbar)^2} \int d\mathbf{R}(s) \\ &\times \frac{\partial}{\partial \phi(s)} \langle J[\mathbf{R}(s), \mathbf{P}(s), s; \mathbf{R}_i, \mathbf{P}_i, 0] \rangle_{a, t} \frac{\partial}{\partial \phi(s)} \langle J[\mathbf{R}_f, \mathbf{P}_f, t; \mathbf{R}(s), \mathbf{P}(s), s] \rangle_{a, t}. \quad (74) \end{aligned}$$

Let us study two distinct examples. The first is the case of homogeneous driving, where

$$D(\mathbf{P}_i, \mathbf{R}_i) = D(\mathbf{P}_i) = \sum_{\nu} \exp(i\nu\phi_i) D_{\nu}(\mathbf{P}_i). \quad (75)$$

The width squared from the ν th mode is then $\nu^2 \lambda_{\text{F}}^2$ for a rotation-invariant system. In a path integral calculation of the impurity scattering integral in the Boltzmann equation the well-known factor $1 - \cos(\nu\theta)$ factor originates from an average of $2 \sin^2[\frac{1}{2}\theta\partial/\partial\phi(s)]$, so as expected it is the width $\nu\lambda_{\text{F}}$ which sets the boundary between the small-angle scattering regime, where the vertex corrections given by $\cos(\nu\theta)$ are important, and a large-angle scattering regime, where the vertex correction averages out. Notice that for an elastic scattering system this averaging has to be performed via the boundary conditions, while in general an interacting system is self-averaging.

As another example we consider the gedanken experiment typically considered in connection with the interpretation of a single-particle Green's function. A particle is fed into the system at the Fermi surface and in a direction parallel to the x axis. In order to make it a bit more realistic we give the direction a finite spread Δ_i ; i.e., we take $D \propto \exp[(1/2\Delta_i^2)\phi_i^2]$. A time t later we measure the probability that the particle is still moving along the x axis with a directional spread of Δ_f ; i.e., we take $\mathcal{R} \propto \exp[(1/2\Delta_f^2)\phi_f^2]$. The width squared is in this case $\lambda_F^2(1/\Delta_i^2 + 1/\Delta_f^2)$. In particular the width diverges as the angular spread to the power -1 .

B. First-order non-time-local correction

Let us move on to discuss the first-order correction to the width squared due to the non-time-local part of the interaction $\Sigma_{\mu', \mu'' = \pm} \mu' \mu'' \int_s^t ds' \int_0^s ds'' (i/\hbar) L_{\text{int}}$:

$$\begin{aligned} \Delta\left(\frac{\hbar}{q_{\perp}^c}\right)^2 &= \lambda_F^2 \mathcal{N} \frac{i}{\hbar} \int \frac{d\mathbf{P}_i}{(2\pi\hbar)^2} \int d\mathbf{R}_f \int \frac{d\mathbf{P}_i}{(2\pi\hbar)^2} \int d\mathbf{R}_i \mathcal{R}(\mathbf{P}_f, \mathbf{R}_f) D(\mathbf{P}_i, \mathbf{R}_i) \int \frac{d\mathbf{P}(s)}{(2\pi\hbar)^2} \int d\mathbf{R}(s) \\ &\times \sum_{\mu', \mu'' = \pm} \mu' \mu'' \int_s^t ds' \int_0^s ds'' \left\langle \left\langle \left\langle \left\langle L_{\text{int}}(s', \mu'; s'', \mu'') \frac{\partial}{\partial \phi(s)} \exp\left(\frac{i}{\hbar} S_0[\mathbf{R}(s), \mathbf{P}(s), s; \mathbf{R}_i, \mathbf{P}_i, 0]\right) \right. \right. \right. \\ &\times \left. \left. \left. \frac{\partial}{\partial \phi(s)} \exp\left(\frac{i}{\hbar} S_0[\mathbf{R}_f, \mathbf{P}_f, t; \mathbf{R}(s), \mathbf{P}(s), s]\right) \right\rangle \right\rangle \right\rangle_{[0, s]} \right\rangle_{[s, t]}. \end{aligned} \quad (76)$$

The contribution from the expansion of the normalization \mathcal{N} , has been neglected since it turns out to cancel a term in the expansion of Eq. (76), which we anyway neglect in our discussion below. The dominant contribution is due to the transverse long-wavelength fluctuations, where $\mathbf{q} \cdot \Delta \mathbf{r} \ll 1$. To lowest order in the interaction the extra contribution from L_{int} , Eq. (58), to the width squared is then

$$\begin{aligned} \Delta\left(\frac{\hbar}{q_{\perp}^c}\right)^2 &= \mathcal{N} \int \frac{d\mathbf{P}_f}{(2\pi\hbar)^2} \int d\mathbf{R}_f \int \frac{d\mathbf{P}_i}{(2\pi\hbar)^2} \int d\mathbf{R}_i \mathcal{R}(\mathbf{P}_f, \mathbf{R}_f) D(\mathbf{P}_i, \mathbf{R}_i) \frac{1}{p_F^2} \int \frac{d\mathbf{P}(s)}{(2\pi\hbar)^2} \int d\mathbf{R}(s) \\ &\times \frac{1}{\hbar} \int_s^t ds' \int_0^s ds'' \int \frac{d\mathbf{q}}{(2\pi\hbar)^2} \int \frac{d\omega}{2\pi} \coth\left(\frac{\hbar\omega}{2k_B T}\right) \text{Im} D_{\perp\perp}(\mathbf{q}, \omega) \exp[-i\omega(s' - s'')] \\ &\times \left\langle \left\langle \left\langle \left\langle \frac{1}{\hbar} \mathbf{q} \cdot \Delta \mathbf{r}(s') \left[\frac{1}{q} \mathbf{q} \times \dot{\mathbf{R}}(s') \right] \cdot \left[\frac{1}{q} \mathbf{q} \times \dot{\mathbf{R}}(s'') \right] \frac{1}{\hbar} \mathbf{q} \cdot \Delta \mathbf{r}(s'') \times \frac{\partial}{\partial \phi(s)} \right. \right. \right. \\ &\times \exp\left(\frac{i}{\hbar} S_0[\mathbf{R}(s), \mathbf{P}(s), s; \mathbf{R}_i, \mathbf{P}_i, 0] + \frac{i}{\hbar} \frac{\partial \varepsilon_{\mathbf{p}(s)}}{\partial \mathbf{p}(s)} \cdot \mathbf{q}(s - s'')\right) \frac{\partial}{\partial \phi(s)} \\ &\times \exp\left(\frac{i}{\hbar} S_0[\mathbf{R}_f, \mathbf{P}_f, t; \mathbf{R}(s), \mathbf{P}(s), s] + \frac{i}{\hbar} \frac{\partial \varepsilon_{\mathbf{p}(s)}}{\partial \mathbf{p}(s)} \cdot \mathbf{q}(s' - s)\right) \left. \left. \left. \right\rangle \right\rangle \right\rangle_{[0, s]} \right\rangle_{[s, t]}. \end{aligned} \quad (77)$$

To lowest order we will neglect the effect of $S_{\text{int}}(s, 0)$ and $S_{\text{int}}(t, s)$ inside the path integrals $\langle \langle \rangle \rangle_{[0, s]}$ and $\langle \langle \rangle \rangle_{[s, t]}$, respectively. The product $\Delta \mathbf{r}(s') \Delta \mathbf{r}(s'')$ we will replace by its average value. The most important contribution comes from the longitudinal part of $\Delta \mathbf{r}(s') \Delta \mathbf{r}(s'')$. Furthermore, it turns out that the relevant time scale in the above integral, $\hbar(v_F q)^{-1}$, for the important fluctuations is smaller than the correlation time of $\Delta \mathbf{r}(s') \Delta \mathbf{r}(s'')$, which is of the order $\hbar(k_B T)^{-1} \hbar g_1 p_F^2 / v_F p_F \approx \hbar(k_B T)^{-1}$, so we replace $\Delta \mathbf{r}(s') \Delta \mathbf{r}(s'')$ by its equal-time longitudinal average value $\frac{1}{3} \hbar^2 v_F^2 / (k_B T)^2$; see Appendix C. Within these approximations the correction to the width squared is

$$\begin{aligned} \Delta\left(\frac{\hbar}{q_{\perp}^c}\right)^2 &= \frac{1}{6} \frac{\hbar^2 v_F^2}{(k_B T)^2} \frac{1}{p_F^2} \int_0^\infty \frac{dq}{2\pi\hbar} q \int \frac{d\omega}{2\pi} \\ &\times \coth\left(\frac{\hbar\omega}{2k_B T}\right) [-\text{Im} D_{\perp\perp}(\mathbf{q}, \omega)]. \end{aligned} \quad (78)$$

For the composite fermions the spectral function

$$-\text{Im} D_{\perp\perp}(\mathbf{q}, \omega) \approx 2\pi\hbar \frac{q}{p_F} \frac{\omega}{\omega^2 + (g_{\eta} q^{1+\eta})^2},$$

Eq. (29), so the width squared is approximately

$$\frac{\hbar^2}{q_{\text{cl}}^2} \approx \lambda_{\text{F}}^2 \left[1 + \left(\frac{v_{\text{F}} p_{\text{F}}}{2k_{\text{B}} T} \right)^{(2\eta-1)/(1+\eta)} \right. \\ \left. \times \left(\frac{v_{\text{F}} p_{\text{F}}}{2\hbar g_{\text{F}} p_{\text{F}}^{\eta+1}} \right)^{3/(1+\eta)} 2^{3/(1+\eta)} \frac{1}{9} \text{csc} \frac{3\pi}{2(1+\eta)} \right]. \quad (79)$$

Here $\eta=1$ corresponds to our prime example, the Coulomb interaction

$$\frac{\hbar^2}{q_{\text{cl}}^2} \approx \left(\frac{\hbar v_{\text{F}}}{2k_{\text{B}} T} \lambda_{\text{F}}^3 \right)^{1/2} \left(\frac{v_{\text{F}} p_{\text{F}}}{2\varepsilon_{\text{C}}} \right)^{3/2} \frac{4}{9} \left(\frac{4}{\pi} \right)^{3/4}. \quad (80)$$

We expect the Fermi energy $\frac{1}{2}v_{\text{F}}p_{\text{F}}$ to be of the same order of magnitude as the energy scale set by the interaction i.e., the average Coulomb energy $\varepsilon_{\text{C}} = (e^2/4\pi\epsilon)\sqrt{n}$. The factor $(v_{\text{F}}p_{\text{F}}/2\varepsilon_{\text{C}})^{3/2} (4/\pi)^{3/4}$ is thus of order 1 and the width is approximately given by the geometric mean of the Fermi wavelength and the de Broglie wavelength.

In perturbative calculations of the self-energy the cutoff is often formulated in terms of a frequency cutoff. The frequency cutoff corresponding to Eq. (80) is

$$\omega_{\text{cut}} = g_1 q_{\text{cl}}^2 \approx \hbar^{-1} \sqrt{\varepsilon_{\text{C}} k_{\text{B}} T} \frac{9}{4} \left(\frac{\pi}{4} \right)^{5/4} \left(\frac{2\varepsilon_{\text{C}}}{v_{\text{F}} p_{\text{F}}} \right)^2. \quad (81)$$

We want to emphasize that this transverse cutoff is considerably larger than the longitudinal cutoff, which in Appendix C is found to be $\sqrt{3}\hbar^{-1}k_{\text{B}}T$. In the next section we show that in the case of Coulomb interaction the effective mass depends logarithmically on ω_{cut} , so the temperature dependence of ω_{cut} and thus the inverse width squared are $\sqrt{T}|\ln T|^2$, which tends much slower to zero than the more conventional linear temperature scaling.⁵

V. EFFECTIVE MASS

In the last section we calculated the width between the forward and backward path, \hbar/q_{cl} , Eq. (80), and we hinted at how this width might play the role of an effective cutoff in the calculation of the effective mass. In this section we elaborate a bit more on this connection.

Experimentally the effective mass of a particle is determined by simultaneously measuring the momentum and the velocity. In a Fermi liquid the momentum is typically the Fermi momentum. The velocity measurement typically involves measuring how far the particle has propagated in a known time span t . This is a little bit at odds with the semi-

classical limit, where the external fields are considered to be smooth, so it is hard to define, yet measure, distances shorter than a transport mean-free-path length. This is necessary in a mass measurement, since the time t must be comparable to or smaller than the transport time to make the path reasonably well defined. Having said this one might hope for an intermediate regime, where it is possible to at least put some limits on the effective mass, while still maintaining the semiclassical line of thought.

One attempt at this is done by Willett, West, and Pfeiffer,¹² who uses a surface acoustic wave (SAW) to generate an external electric field with a wavelength comparable to the mean free path. They then apply an external magnetic field and look for resonances between the cyclotron radius and the wavelength of the SAW field. In order to observe these resonances the cyclotron frequency cannot be orders of magnitude larger than frequency of the SAW wave.²¹ This leads to an upper estimate of the elapsed time and thus the effective mass.

We will denote this sort of experiment as a semiclassical mass determination, since the forward and backward paths together perform the cyclotron motion. This is in contrast to experiments probing the single-particle energy spectrum, like Shubnikov–de Haas measurements. In such experiments according to the Bohr-Sommerfeld quantization scheme it is the phase of a single path that causes the resonance.

Below we show that in the semiclassical experiments the width between the two paths or in other words the enclosed area acts as an effective cutoff on the contribution from the magnetic field fluctuations to the effective mass. In the Bohr-Sommerfeld resonance scheme the cutoff is supplied by the area enclosed by the path itself.²² This area is much bigger than the semiclassical area and the cutoff is correspondingly smaller.

In order to extract the semiclassical mass from the path integral description it is in principle necessary to analyze the concrete experiment in detail. Luckily it turns out that in the semiclassical limit it is possible to come up with an approximately local relation between the velocity and the momentum. This implies that the effective mass is almost independent of the concrete mass measuring experiment in the semiclassical limit.

For a noninteracting system the derivative of the action with respect to the momentum yields the Hamilton equation for the velocity $\delta S_0/\delta p(s) = \dot{r} - \partial H/\partial p = 0$. For the two-path action it is the derivative with respect to the difference momentum $\Delta\mathbf{p}(s)$:

$$\frac{\partial S}{\partial \Delta\mathbf{p}(s)} = \dot{\mathbf{R}}(s) - \frac{1}{m} \mathbf{P}(s) - \int_s^t ds' \int \frac{d\mathbf{q}}{(2\pi\hbar)^2} \int \frac{d\omega}{2\pi} D_{\perp\perp}(\mathbf{q}, \omega) \\ \times \exp\left\{ \frac{i}{\hbar} \mathbf{q} \cdot [\mathbf{R}(s') - \mathbf{R}(s)] \right\} \exp[-i\omega(s' - s)] \times \sum_{\mu, \mu' = \pm} \mu \mu' \exp\left\{ \frac{i}{\hbar} \mu \frac{1}{2} \mathbf{q} \cdot [\Delta\mathbf{r}(s') - \Delta\mathbf{r}(s)] \right\} \frac{1}{q} \\ \times \left[\dot{\mathbf{R}}(s') + \frac{1}{2} \mu' \Delta\dot{\mathbf{r}}(s') \right] \times \frac{\partial}{\partial \mathbf{P}} \left\{ \frac{1}{2} - w \left[\mathbf{P}(s^+) + \mu \frac{1}{2} \Delta\mathbf{p}(s^+) \right] \frac{1}{mq} \mathbf{q} \times \left[\mathbf{P}(s) + \mu \frac{1}{2} \Delta\mathbf{p}(s) \right] \right\}, \quad (82)$$

which yields the relation between the velocity and the momentum, at least in a stationary phase approximation where $\partial S/\partial \Delta \mathbf{p}(s)=0$. On average the stationary phase condition is exact, i.e.,

$$\left\langle \left| \frac{\partial S}{\partial \Delta \mathbf{p}(s)} \right| \right\rangle = 0. \quad (83)$$

Notice that after performing the integral over $\mathbf{R}(s)$ and $\Delta \mathbf{r}(s)$ inside the average as defined in Eq. (68) we have $\Delta \mathbf{p}(s^+) = \Delta \mathbf{p}(s) - \mathbf{q}$ and $\mathbf{P}(s^+) = \mathbf{P}(s) - \frac{1}{2} \mu \mathbf{q}$. The terms with $\mu = \mu'$ in Eq. (82), i.e., those terms where the interaction connects either the forward or backward path with itself, give rise to that part of the renormalization of the effective mass that is attributed to the self-energy in the usual perturbative treatments. This renormalization mainly depends on the local properties of a single path and it is thus independent of which pair of paths is relevant in the given experiment. On the other hand, the terms with $\mu \neq \mu'$ in Eq. (82), which are associated with the vertex corrections, depend on both the forward and backward paths. The renormalization due to these terms thus depends on the exact experiment. For instance in an interference experiment the two paths are almost independent and the vertex contribution is expected to vanish due to destructive interference in the phases associated with each path individually. In the semiclassical limit, where the forward and backward paths typically are close to each other, it is fortunately possible to derive a universal expression for the renormalization of the mass. Below it will be shown that the interaction modes with a wavelength longer than the separation between the paths does not renormalize the mass, while the renormalization due to the interaction modes with a shorter wavelength is given by the usual self-energy contribution. A possible interpretation is that the particle cannot distinguish between the field from a long-wavelength interaction and an external potential.

For simplicity we neglect the extra terms arising from the velocity dependence of the force; i.e., we treat the interaction as scalar. With this small neglect the average of the velocity, denoted $\mathbf{v}_F = \langle |\dot{\mathbf{R}}| \rangle$, is

$$\begin{aligned} \mathbf{v}_F &= \frac{1}{m} \mathbf{p}_F - \int_s^t ds' \int \frac{d\mathbf{q}}{(2\pi\hbar)^2} \int \frac{d\omega}{2\pi} v_F m^{-1} p_F D_{\perp\perp}(\mathbf{q}, \omega) \\ &\times e^{-i\omega(s'-s)} \left\langle \exp\left(\frac{i}{\hbar} \mathbf{q} \cdot [\mathbf{R}(s') - \mathbf{R}(s)]\right) 2i \right. \\ &\times \sin\left[\frac{1}{2\hbar} \mathbf{q} \cdot \Delta \mathbf{r}(s')\right] \sum_{\mu=\pm} \mu \exp\left(-\frac{i}{\hbar} \mu \frac{1}{2} \mathbf{q} \cdot \Delta \mathbf{r}(s)\right) \\ &\left. \times \frac{\partial}{\partial \mathbf{P}} w\left[\mathbf{P}(s) + \mu \frac{1}{2} \Delta \mathbf{p}(s) - \mu \mathbf{q}\right] \right\rangle, \quad (84) \end{aligned}$$

where in the first term it is used that the average momentum

$\langle |\mathbf{P}| \rangle$ is the Fermi momentum $\mathbf{p}_F = p_F v_F^{-1} \mathbf{v}_F$ (see Appendix C). At least for a noninteracting system $\Delta \mathbf{p}(s)$ is of the order of the typical wave vector of the external force field. Our interest is the semiclassical limit, so we will assume $\Delta \mathbf{p}(s) \cdot \mathbf{p}_F < k_B T$ and neglect its contribution; i.e., we put $\Delta \mathbf{p}(s) = 0$. The average velocity \mathbf{v}_F is then

$$\begin{aligned} \mathbf{v}_F &= \frac{1}{m} \mathbf{p}_F - \int_s^t ds' \int \frac{d\mathbf{q}}{(2\pi\hbar)^2} \int \frac{d\omega}{2\pi} v_F m^{-1} p_F D_{\perp\perp}(\mathbf{q}, \omega) \\ &\times \exp\left(\frac{i}{\hbar} \mathbf{q} \cdot [\mathbf{R}(s') - \mathbf{R}(s)]\right) \exp[-i\omega(s'-s)] \\ &\times \left\langle \left| 2 \sin\left[\frac{1}{2\hbar} \mathbf{q} \cdot \Delta \mathbf{r}(s')\right] \right. \right. \\ &\times \left\{ \sin\left[\frac{1}{2\hbar} \mathbf{q} \cdot \Delta \mathbf{r}(s)\right] \sum_{\mu=\pm} \frac{\partial}{\partial \mathbf{P}} w[\mathbf{P}(s) - \mu \mathbf{q}] \right. \\ &\left. \left. + i \cos\left[\frac{1}{2\hbar} \mathbf{q} \cdot \Delta \mathbf{r}(s)\right] \sum_{\mu=\pm} \mu \frac{\partial}{\partial \mathbf{P}} w[\mathbf{P}(s) - \mu \mathbf{q}] \right\} \right\rangle. \quad (85) \end{aligned}$$

We are mainly interested in the long-wavelength limit, so we will neglect terms of order q^2 compared to terms of order q . We also assume that both $|\mathbf{P}(s)|$ and the corresponding velocity $|\dot{\mathbf{R}}(s)|$ are distributed evenly around the Fermi surface. In this approximation the second term inside the parentheses can be neglected. We assume that the relative variation in the velocity around the Fermi velocity \mathbf{v}_F is small; i.e., the temperature is much smaller than the Fermi energy. It is thus possible to approximate the path locally by a straight line, i.e., $\mathbf{R}(s') - \mathbf{R}(s) \approx \mathbf{v}_F(s'-s)$. In the long-wavelength limit the transverse part of $\mathbf{q} \cdot \Delta \mathbf{r}$ is bigger than the longitudinal part. The averaging procedure is approximated by independently carrying out the longitudinal and transverse averages. In the longitudinal average we use the time-local approximation outlined in Appendix C, so the longitudinal average of $\sum_{\mu=\pm} -(\partial/\partial \mathbf{P}) w[\mathbf{P}(s) - \mu \mathbf{q}]$ is approximately

$$\begin{aligned} \left\langle \left| \sum_{\mu=\pm} -\frac{\partial}{\partial \mathbf{P}} w[\mathbf{P}(s) - \mu \mathbf{q}] \right| \right\rangle &\approx \mathbf{v}_F \int_{-\infty}^{\infty} d\varepsilon \frac{\partial}{\partial \varepsilon} f_F(\varepsilon) \\ &\times \sum_{\mu=\pm} \frac{\partial}{\partial \varepsilon} f_F(\varepsilon - \mu \mathbf{v}_F \cdot \mathbf{q}), \quad (86) \end{aligned}$$

where we have used $\mathbf{v}_F = \partial \varepsilon_{\mathbf{P}} / \partial \mathbf{P}$. In the transverse average we approximate

$$\begin{aligned} 2 \sin\left[\frac{1}{2\hbar} \mathbf{q} \cdot \Delta \mathbf{r}(s')\right] \sin\left[\frac{1}{2\hbar} \mathbf{q} \cdot \Delta \mathbf{r}(s)\right] \\ \approx \left\{ 1 - \cos\left[\frac{1}{\hbar} \mathbf{q} \cdot \Delta \mathbf{r}(s)\right] \right\} \approx \left[1 - \exp\left(-\frac{1}{2} q_{\perp}^{-2} q^2\right) \right], \quad (87) \end{aligned}$$

with the result that the effective mass m^* , defined as $\mathbf{p}_F = m^* \mathbf{v}_F$, is given by

$$\begin{aligned} \frac{1}{m^*} \approx \frac{1}{m} & \left\{ 1 + 2v_F^2 \int_{-\infty}^{\infty} ds' \int \frac{d\mathbf{q}}{(2\pi\hbar)^2} \int \frac{d\omega}{2\pi} \text{Re} D_{\perp\perp}(\mathbf{q}, \omega) \right. \\ & \times \left[1 - \exp\left(-\frac{1}{2} q_{\perp}^{-2} q^2\right) \right] \times \exp\left(\frac{i}{\hbar} \mathbf{q} \cdot \mathbf{v}_F (s' - s)\right) \\ & \times \exp[-i\omega(s' - s)] \int_{-\infty}^{\infty} d\varepsilon \\ & \left. \times \frac{\partial}{\partial \varepsilon} f_F(\varepsilon) \frac{\partial}{\partial \varepsilon} f_F(\varepsilon - \mathbf{v}_F \cdot \mathbf{q}) \right\}. \end{aligned} \quad (88)$$

Apart from the extra factor of $[1 - \exp(-\frac{1}{2} q_{\perp}^{-2} q^2)]$ this relation is identical to the usual relation derived from the self-energy. Thus the contribution from the short-wavelength modes of the interaction to the renormalization of the mass is identical to the standard result, while the long-wave length modes do not contribute. The cross over is at the width $\hbar q_{\perp}^{-1}$ and this is why we above have alluded to the width as an effective cutoff.

When the long-wavelength expression for the propagator $D_{\perp\perp}$, Eq. (29), in the case of Coulomb interaction, is inserted into Eq. (88) the integral is ultraviolet divergent. We use p_F as an approximative upper cutoff and get the effective mass

$$m^* \approx m + m \frac{4}{\pi^{3/2}} \frac{p_F^2}{2m} \frac{1}{\varepsilon_C} \ln \frac{p_F^2}{2q_{\perp}^2}. \quad (89)$$

For composite fermions the renormalization of the mass is quite strong, so the effective mass

$$m^* \approx \frac{2}{\pi^{3/2}} \frac{p_F^2}{\varepsilon_C} \ln \frac{p_F^2}{2q_{\perp}^2} \quad (90)$$

is approximately independent of the bare mass m .

The prefactor

$$m \frac{4}{\pi^{3/2}} \frac{p_F^2}{2m} \frac{1}{\varepsilon_C} \approx 1.9 m, \quad (91)$$

where $\varepsilon_C = (e^2/4\pi\epsilon) \sqrt{n}$ is the average Coulomb energy and we as a typical values¹² have used $n = 1.6 \times 10^{15} \text{ m}^{-2}$, $\epsilon = 13\epsilon_0$, and $m \approx 0.067m_0$, where ϵ_0 and m_0 are the vacuum values of the permittivity and the electron mass, respectively.

The argument of the logarithm is given by Eq. (79), so self-consistently the effective mass is

$$\begin{aligned} \frac{m^*}{m} \approx 1 + \frac{4}{\pi^{3/2}} \frac{p_F^2}{2m} \frac{1}{\varepsilon_C} \ln \left[\frac{1}{2} + \frac{\pi^{9/4}}{18} \sqrt{\frac{\varepsilon_C}{k_B T}} \right. \\ \left. \times \left(\frac{4}{\pi^{3/2}} \frac{p_F^2}{2m} \frac{1}{\varepsilon_C} \right)^2 \left(\frac{m}{m^*} \right)^2 \right] \approx 6.3, \end{aligned} \quad (92)$$

where the estimate is valid at a temperature of $\sim 0.13 \text{ K}$. The cutoff at this temperature happens at distances $\hbar q_{\perp}^{-1} \approx 6\lambda_F \approx 40 \text{ nm}$.

As expected the semiclassical effective mass, Eq. (92), is smaller than the mass deduced from the activation energy and Shubnikov–de Haas measurements,¹² actually about half as big. In the experiment by Willett *et al.*¹² the estimate, Eq. (92), leads to a cyclotron frequency that at the secondary resonance position is about twice as big as the used SAW frequency, 10.7 GHz. A naive argument¹² leads to the expectation that the secondary resonance can only be observed if the cyclotron frequency is much bigger than the SAW frequency. It is thus a matter of taste whether an ‘‘apparent inconsistency’’ between the geometric resonance experiments and the size of the effective mass arises. We here want to stress the word ‘‘apparent’’ since Mirlin and Wölfle²¹ have shown that the secondary resonance can be observed even when the cyclotron frequency equals the SAW frequency.

VI. CONCLUSION

In this paper we have developed a single-particle path integral description of transport for composite fermions, thereby extending the Caldeira-Leggett formalism for a single particle interacting with an environment. Here the environment is the Chern-Simons gauge field. Our approach is modeled over the work by Golubev and Zaikin on the electron gas in zero magnetic field. As in the Caldeira-Leggett formalism there is not one, but two paths in the path integral, one propagating forward in time and one backward. A new feature compared to the traditional Caldeira-Leggett formalism is that the distribution function enters into the real part of the single-particle action. The imaginary part of the action is determined by the fluctuation spectrum, just as in the Caldeira-Leggett formalism. Actually, the imaginary part is so familiar that its effect previously has been included by hand in calculations based on the use of paths.^{14,22} In the second half of this work the formalism was applied to the calculation of the effective mass, a quantity that depends crucially on the real part of the action. This is a major result, since previously it has not been possible to discuss effects related to the renormalization of the propagation properties including the mass within a path integral approach.

Furthermore, for CF’s the effective mass turns out to depend on the actual experiment. In a perturbative description of the CF’s the nonuniversality of the effective mass is attributed to the fact that the long-wave length magnetic field fluctuations give rise to an infrared divergence of the self-energy and the vertex corrections, separately. In a transport measurement these divergences are known to cancel. In our formalism the infrared divergence does not appear because the vertex and self-energy contributions are never split. Not only is the infrared divergence absent, but the renormalization in the infrared limit is negligible. Furthermore, in the semiclassical limit the effective mass is universal, independent of the actual semiclassical experiment, just as if there were some underlying quasiparticles. Recently, various for-

mulations based on dipolar quasiparticles have emerged.^{9–11} Using such an approach Shankar and Murthy noticed that in a particular approximation, where some of the constraints are neglected, the system is not compressible at half filling.⁹ This very cautiously phrased suggestion has been challenged by Halperin and Stern.^{23,24} Our formulation naturally leads to a compressible system and thus supports the scenario put forward by Halperin and Stern.

By working in a real-space formulation our results are obtained in a natural way, since the magnetic interaction acts via fluxes, i.e., areas in real space. In the path integral formalism the size of the particle is defined as the separation between the forward and backward propagating paths. Only those magnetic field fluctuations with a wavelength shorter than this particle size contribute noticeably to the renormalization of the mass. In GaAs at $T \sim 0.13$ K we find the mass renormalization due to the magnetic field fluctuations to be roughly 6 times the bare mass. Furthermore, we have shown that in the semiclassical limit the separation between the two paths, i.e., the size of the particle, scales as $T^{-1/4} |\ln T|^{-1}$. This implies that the effective cutoff in frequency scales as roughly $T^{1/2} |\ln T|^2$. As usual the mass scales approximately with the logarithm of the frequency cutoff.

Experimentally it is not very easy to measure the mass in the semiclassical limit. The best one can do is to put an upper limit on the size of the mass.^{12,25} Our result is well within the limit derived from the experiments by Willett *et al.*^{12,21} In standard Fermi liquid theory the mass in the semiclassical regime is identical to the mass obtained from single-particle probes like the Shubnikov–de Haas oscillations. For composite fermions this is not the case, and furthermore the mass derived using standard Fermi liquid fitting formulas differs from experiment to experiment.^{11,26–28} In the path integral formulation single-particle probes are associated with pairs of paths, where the forward and backward paths differ qualitatively,^{14,29} e.g., one of them loops an extra time around a cyclotron orbit or the two paths pass through different slits in a double-slit experiment. The gauge field singular contribution to the effective mass is thus no longer determined by the semiclassical size of the particle, but rather the nonuniversal geometry of the particular single-particle probe, e.g., the area of the extra cyclotron orbit. The effective mass thus depends on the particular geometry of the experiment. However, since the single-particle cutoff areas are larger than the corresponding semiclassical area, the semiclassical mass is smaller than the one obtained from a single-particle probe. Indeed in the experiments by Willett *et al.*¹² it turns out to be a factor of 2 smaller.

The linear response regime around equilibrium is the major focus of our work, but the formalism is still valid out of equilibrium. In the noninteracting limit, as shown in Fig. 3, the single-particle path integral technique possesses two important features: (i) the effect of the nonequilibrium distribution function is gathered completely within the stimulus, and (ii) the propagation out of equilibrium is identical to the propagation in equilibrium. These features are shared with other out-of-equilibrium techniques, e.g., the Boltzmann equation and the Landauer–Büttiker description of transport through a noninteracting part of the system. In the case of an

interacting system (i) and (ii) no longer holds rigorously, but in some cases they still are expected to be a good starting point. The path integral formulation is then well suited to study the lowest-order corrections due to the interactions. An example of a system where this approach might be useful is a standard finite source drain measurement. For such a system the most important effect is presumably due to an out-of-equilibrium stimulus. For a quantum point contact this has already been explored using Green’s functions.³⁰ However, in general for an interacting system the propagation properties are expected to be renormalized too. Within the path integral formalism this should be possible to treat, especially if the nonequilibrium distribution function turns out, at least approximately, to be a translated Fermi distribution with an increased temperature. To our knowledge nobody has attempted this yet.

The single-particle path integral technique is thus a promising technique for the description of interacting systems both in and out of equilibrium; especially, it may prove useful for studying systems out of equilibrium not easily treated by other methods.

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APPENDIX A: EQUATIONS OF MOTION

In this appendix we are going to derive differential equations for the composite fermion contribution to the effective action $S_{\text{eff}}[a; t]$ and the density matrix $\rho_{ac}(A_{\text{eff}} - a)$ in the presence of the Chern-Simons field $A_{\text{eff}} - a$ by differentiating the composite fermion part of Eqs. (18) and (15). In order to differentiate quantities in a fixed gauge field background at time s it is necessary to let the Chern-Simons field a and the Grassmann fields $\bar{\psi}$ and ψ be defined on C_t , where $t > s$. This is achieved by considering the generalized problem, where the Hamiltonian is artificially put equal to zero for times $s' > s$. All the steps leading to the effective action for composite fermions S_{CF} , Eq. (10), can also be carried through in this generalized setting. The only difference is that in all the terms that involve the Hamiltonian the contour integration is effectively limited to C_s . We denote the actions in this generalized case by their usual symbol, but with $s; t$ as the temporal variable, so the effective action for the composite fermions is

$$S_{\text{CF}}[\bar{\psi}, \psi, a; s; t] = S_{\text{CS}}[a; s; t] - e \int_{C_t} d\sigma' \int d\mathbf{r} a_0(\mathbf{r}, \sigma') n + S_{\text{ComFer}}[\bar{\psi}, \psi, a; s; t], \quad (\text{A1})$$

where

$$S_{\text{ComFer}}[\bar{\psi}, \psi, a; s; t] = \int_{C_t} d\sigma' \int d\mathbf{r} \bar{\psi}(\mathbf{r}, \sigma') i\hbar \frac{\partial}{\partial \sigma'} \psi(\mathbf{r}, \sigma') - \int_{C_s} d\sigma' H_0[A_{\text{eff}} - a]. \quad (\text{A2})$$

The terms that depend on the composite fermion Grassmann variables are gathered in $S_{\text{ComFer}}[\bar{\psi}, \psi, a; s; t]$ and it is its contribution we are going to consider in this appendix.

Notice also that when $s=t$ the generalized problem is identical to the original problem, so for all actions is $S(t) = S(s; t)|_{s=t}$.

1. Effective action

The composite fermion contribution to the effective action $S_{\text{eff}}[a; s; t]$ is

$$S_{\text{eff}}[a; s; t] - S_{\text{CS}}[a; s; t] + e \int_{-\infty}^t ds' \int d\mathbf{r} \Delta a_0(\mathbf{r}, s') n = \frac{\hbar}{i} \ln \langle\langle 1 \rangle\rangle, \quad (\text{A3})$$

where $\Delta a_0(s') = a_0(s', +) - a_0(s', -)$ is the difference field and the shorthand notation

$$\langle\langle X \rangle\rangle = \int \mathcal{D}\bar{\psi} \int \mathcal{D}\psi \exp\left(\frac{i}{\hbar} S_{\text{ComFer}}[\bar{\psi}, \psi, a; s; t]\right) X \quad (\text{A4})$$

for the Grassmann integral has been introduced. The derivative of Eq. (A3) is

$$\partial_s \{\text{Eq. (A3)}\} = - \langle\langle 1 \rangle\rangle^{-1} \langle\langle H_0[A_{\text{eff}}(s) - a(s, +)] - H_0[A_{\text{eff}}(s) - a(s, -)] \rangle\rangle. \quad (\text{A5})$$

The right-hand side of Eq. (A5) is equal to the expectation value of $-\{H_0[A_{\text{eff}}(s) - a(s, +)] - H_0[A_{\text{eff}}(s) - a(s, -)]\}$ in the presence of the contour gauge field $A_{\text{eff}} - a$. It can be expressed in terms of the density matrix as

$$\partial_s \{\text{Eq. (A3)}\} = - \int d\mathbf{r} [e \Delta a_0(\mathbf{r}, s) \rho_{ac}(A_{\text{eff}} - a; \mathbf{r}, \mathbf{r}; s) + \Delta \mathbf{a}(\mathbf{r}, s) \cdot \mathbf{j}_a(A_{\text{eff}} - a; \mathbf{r}, s)], \quad (\text{A6})$$

where the current in the presence of the gauge field $A_{\text{eff}} - a$, $\mathbf{j}_a(A_{\text{eff}} - a; \mathbf{r}, s)$, is defined in Eq. (19). Equation (20) then follows upon integration using the conventional assumption of adiabatic turning on of the gauge field.

2. Density matrix

The density matrix in the presence of the Chern-Simons gauge field ρ_{ac} is defined in Eq. (15). At a time $s < t$ the action $S_{\text{CF}}[\bar{\psi}, \psi, a; s]$ can be replaced by $S_{\text{ComFer}}[\bar{\psi}, \psi, a; s; t]$. In terms of the above shorthand notation,

$$\rho_{ac}(\mathbf{r}, \mathbf{r}', s) = \langle\langle 1 \rangle\rangle^{-1} \langle\langle \bar{\psi}(\mathbf{r}', s) \psi(\mathbf{r}, s) \rangle\rangle. \quad (\text{A7})$$

When s is varied both the numerator and denominator vary:

$$i\hbar \partial_s \rho_{ac}(\mathbf{r}, \mathbf{r}', s) = \langle\langle 1 \rangle\rangle^{-1} i\hbar \partial_s \langle\langle \bar{\psi}(\mathbf{r}', s) \psi(\mathbf{r}, s) \rangle\rangle - \langle\langle 1 \rangle\rangle^{-1} \langle\langle \bar{\psi}(\mathbf{r}', s) \psi(\mathbf{r}, s) \rangle\rangle \times i\hbar \partial_s \langle\langle 1 \rangle\rangle. \quad (\text{A8})$$

In the above subsection, the derivative of the denominator $i\hbar \partial_s \langle\langle 1 \rangle\rangle$ was calculated. The contribution from the first term (the numerator) is

$$\langle\langle 1 \rangle\rangle^{-1} i\hbar \partial_s \langle\langle \bar{\psi}(\mathbf{r}', s) \psi(\mathbf{r}, s) \rangle\rangle \quad (\text{A9})$$

$$= \langle\langle 1 \rangle\rangle^{-2} \{ \langle\langle H_0[A_{\text{eff}}(s) - a(s, +)] \bar{\psi}(\mathbf{r}', s) \psi(\mathbf{r}, s) - \bar{\psi}(\mathbf{r}', s) \psi(\mathbf{r}, s) H_0[A_{\text{eff}}(s) - a(s, -)] \rangle\rangle \}. \quad (\text{A10})$$

We here used

$$\langle\langle \bar{\psi}(\mathbf{r}', s + \Delta s) \psi(\mathbf{r}, s + \Delta s) \rangle\rangle = \langle\langle \bar{\psi}(\mathbf{r}', s) \psi(\mathbf{r}, s) \rangle\rangle, \quad (\text{A11})$$

since without a Hamiltonian the system does not evolve. The action S_{ComFer} is quadratic in the Grassmann variables $\bar{\psi}$ and ψ , so Eq. (A10) can be evaluated with the help of Wick's theorem. The Hartree disconnected diagram like pairings exactly cancels the contribution from the variation of the denominator in Eq. (A8), $\langle\langle 1 \rangle\rangle^{-1} \langle\langle \bar{\psi}(\mathbf{r}', s) \psi(\mathbf{r}, s) \rangle\rangle i\hbar \partial_s \langle\langle 1 \rangle\rangle$, so paying due attention to the contour ordering

$$i\hbar \partial_t \rho_{ac}(\mathbf{r}, \mathbf{r}', t) = \int d\mathbf{r}'' \int d\mathbf{r}''' (\delta(\mathbf{r} - \mathbf{r}'') - \rho_{ac}[\mathbf{r}, \mathbf{r}''; t]) H(\mathbf{r}'', \mathbf{r}'''; t, +) \rho_{ac}[\mathbf{r}''', \mathbf{r}'; t] - \int d\mathbf{r}'' \int d\mathbf{r}''' \rho_{ac}[\mathbf{r}, \mathbf{r}''; t] H(\mathbf{r}'', \mathbf{r}'''; t, -) \times (\delta(\mathbf{r}''' - \mathbf{r}') - \rho_{ac}[\mathbf{r}''', \mathbf{r}'; t]). \quad (\text{A12})$$

This is Eq. (31) written out.

APPENDIX B: LINEARIZATION OF THE DENSITY MATRIX DIFFERENTIAL EQUATION

The stimulus in the linearized equation of motion for $\delta\rho_{ac}(A_{\text{eff}} - a; \mathbf{r}, \mathbf{r}', t)$, Eq. (40), $D = D_1 + D_2$ is split into two parts. The explicit form for D_1 , which is independent of $\Delta \mathbf{a}$, is

$$\begin{aligned}
i\hbar D_1(\mathbf{r}, \mathbf{r}', t) = & \left[-\frac{\partial}{\partial \mathbf{r}} \int_{\mathbf{r}'}^{\mathbf{r}} d\mathbf{r}'' \cdot \mathbf{A}_{\text{tot}}(\mathbf{r}'') + \mathbf{A}_{\text{tot}}(\mathbf{r}) \right] \cdot \frac{1}{m} \left[\frac{\hbar}{i} \frac{\partial}{\partial \mathbf{r}} + eA_{\text{eff}}(\mathbf{r}) - e\bar{\mathbf{a}}(\mathbf{r}) \right] \rho_{ac}(\mathbf{r}, \mathbf{r}') \\
& + \left[-\frac{\partial}{\partial \mathbf{r}'} \int_{\mathbf{r}'}^{\mathbf{r}} d\mathbf{r}'' \cdot \mathbf{A}_{\text{tot}}(\mathbf{r}'') - \mathbf{A}_{\text{tot}}(\mathbf{r}') \right] \cdot \frac{1}{m} \left[-\frac{\hbar}{i} \frac{\partial}{\partial \mathbf{r}'} + eA_{\text{eff}}(\mathbf{r}') - e\bar{\mathbf{a}}(\mathbf{r}') \right] \rho_{ac}(\mathbf{r}, \mathbf{r}') \\
& + \rho_{ac}(\mathbf{r}, \mathbf{r}') \frac{\hbar}{2im} \frac{\partial}{\partial \mathbf{r}} \left[-\frac{\partial}{\partial \mathbf{r}'} \int_{\mathbf{r}'}^{\mathbf{r}} d\mathbf{r}'' \cdot \mathbf{A}_{\text{tot}}(\mathbf{r}'') - \mathbf{A}_{\text{tot}}(\mathbf{r}) \right] - \rho_{ac}(\mathbf{r}, \mathbf{r}') \frac{\hbar}{2im} \frac{\partial}{\partial \mathbf{r}'} \\
& \times \left[-\frac{\partial}{\partial \mathbf{r}'} \int_{\mathbf{r}'}^{\mathbf{r}} d\mathbf{r}'' \cdot \mathbf{A}_{\text{tot}}(\mathbf{r}'') - \mathbf{A}_{\text{tot}}(\mathbf{r}') \right] - \int_{\mathbf{r}'}^{\mathbf{r}} d\mathbf{r}'' \cdot \left(\partial_t \mathbf{A}_{\text{tot}} + \frac{\partial \phi_{\text{tot}}}{\partial \mathbf{r}''} \right) \rho_{ac}(\mathbf{r}, \mathbf{r}'). \tag{B1}
\end{aligned}$$

The explicit form of D_2 , which contains all the dependence upon $\Delta \mathbf{a}$, is

$$\begin{aligned}
i\hbar D_2(\mathbf{r}, \mathbf{r}', t) = & \int d\mathbf{r}'' (1 - \rho_{ac})(\mathbf{r}, \mathbf{r}'') \frac{-e\Delta \mathbf{a}(\mathbf{r}'')}{2m} \cdot \left[-\frac{\partial}{\partial \mathbf{r}''} \int_{\mathbf{r}'}^{\mathbf{r}''} d\mathbf{r}''' \cdot \mathbf{A}_{\text{tot}}(\mathbf{r}''') + \mathbf{A}_{\text{tot}}(\mathbf{r}'') \right] \rho_{ac}(\mathbf{r}'', \mathbf{r}') \\
& + \int d\mathbf{r}'' \rho_{ac}(\mathbf{r}, \mathbf{r}'') \frac{-e\Delta \mathbf{a}(\mathbf{r}'')}{2m} \cdot \left[\frac{\partial}{\partial \mathbf{r}''} \int_{\mathbf{r}'}^{\mathbf{r}} d\mathbf{r}''' \cdot \mathbf{A}_{\text{tot}}(\mathbf{r}''') + \mathbf{A}_{\text{tot}}(\mathbf{r}'') \right] (1 - \rho_{ac})(\mathbf{r}'', \mathbf{r}') \\
& - \frac{i}{2\hbar} \int d\mathbf{r}'' \rho_{ac}(\mathbf{r}, \mathbf{r}'') \left[\int_{\mathbf{r}'}^{\mathbf{r}} d\mathbf{r}''' \cdot \mathbf{A}_{\text{tot}}(\mathbf{r}''') - \int_{\mathbf{r}'}^{\mathbf{r}''} d\mathbf{r}''' \cdot \mathbf{A}_{\text{tot}}(\mathbf{r}''') - \int_{\mathbf{r}''}^{\mathbf{r}'} d\mathbf{r}''' \cdot \mathbf{A}_{\text{tot}}(\mathbf{r}''') \right] \\
& \times \frac{-e\Delta \mathbf{a}}{2m} \cdot \left[\frac{1}{2m} \Delta \mathbf{a} \left(\frac{i}{\hbar} \frac{\partial}{\partial \mathbf{r}''} + \mathbf{A}_{\text{eff}}(\mathbf{r}'') \right) - \Delta a_0 \right] \rho_{ac}(\mathbf{r}'', \mathbf{r}') \\
& - \frac{i}{2\hbar} \int d\mathbf{r}'' \rho_{ac}(\mathbf{r}'', \mathbf{r}') \left[\int_{\mathbf{r}'}^{\mathbf{r}} d\mathbf{r}''' \cdot \mathbf{A}_{\text{tot}}(\mathbf{r}''') - \int_{\mathbf{r}'}^{\mathbf{r}''} d\mathbf{r}''' \cdot \mathbf{A}_{\text{tot}}(\mathbf{r}''') - \int_{\mathbf{r}''}^{\mathbf{r}'} d\mathbf{r}''' \cdot \mathbf{A}_{\text{tot}}(\mathbf{r}''') \right] \\
& \times \frac{-e\Delta \mathbf{a}}{2m} \cdot \left[\frac{1}{2m} \Delta \mathbf{a} \left(-\frac{i}{\hbar} \frac{\partial}{\partial \mathbf{r}''} + \mathbf{A}_{\text{eff}}(\mathbf{r}'') \right) - \Delta a_0 \right] \rho_{ac}(\mathbf{r}, \mathbf{r}''). \tag{B2}
\end{aligned}$$

APPENDIX C: THE LONGITUDINAL SEPARATION

In this appendix we are interested in the distribution of the longitudinal separation $(1/P)\mathbf{P} \cdot \Delta \mathbf{r}$ and its conjugate variable, the size of the momentum P . For simplicity we will only consider the case of a scalar interaction.

The first step is to rewrite the free particle action such that the longitudinal and transverse directions appear explicitly.

To this end we use polar coordinates for the momentum:

$$\mathbf{p} = \begin{pmatrix} p \cos \frac{p_\eta}{p} \\ p \sin \frac{p_\eta}{p} \end{pmatrix}, \tag{C1}$$

where p_η/p is the angle with respect to an arbitrarily defined x axis. p_η , loosely speaking the arc length along the Fermi surface, has been chosen as our variable instead of the angle p_η/p in order to avoid a complicated Jacobian. In the scalar approximation the interaction is independent of p_η , which therefore can be integrated out. The free particle action on the forward part of the contour is

$$\int_0^t ds \left(\dot{\mathbf{r}} \cdot \mathbf{p} - \frac{\mathbf{p}^2}{2m} \right). \tag{C2}$$

The stationary phase approximation for p_η reads

$$p_\eta = p \arctan \frac{\dot{y}}{x}. \tag{C3}$$

There are two solutions to this equation corresponding to the momentum and the velocity being parallel or antiparallel. Here we only keep the parallel solution. The resulting stationary phase action on the forward contour is

$$\int_0^t ds \left(pv - \frac{\mathbf{p}^2}{2m} \right),$$

where $v = \sqrt{\dot{x}^2 + \dot{y}^2}$ is the velocity. A similar result holds on the backward path, but with the opposite overall sign. We are interested in discussing the separation between the two paths, so we change representation from the position and momentum along each path individually to the average $\mathbf{R}(s) = \frac{1}{2}[\mathbf{r}(s, +) + \mathbf{r}(s, -)]$ and the difference $\Delta \mathbf{r}(s) = \mathbf{r}(s, +) - \mathbf{r}(s, -)$.

$-\mathbf{r}(s, -)$. In a translation-invariant system one has the choice between using \mathbf{R} or $\dot{\mathbf{R}}$ as the variables in the path integral. Here we choose $\dot{\mathbf{R}}$ and we parametrize it using the velocity $V(s) = |\dot{\mathbf{R}}(s)|$ and the polar angle $V_\phi(s)/V(s)$:

$$\dot{\mathbf{R}}(s) = \begin{pmatrix} V(s) \cos \frac{V_\phi(s)}{V(s)} \\ V(s) \sin \frac{V_\phi(s)}{V(s)} \end{pmatrix} = V(s) \mathbf{n}(s). \quad (\text{C4})$$

The unit vector $\mathbf{n}(s)$ points along the direction of the average propagation. $\mathbf{n}(s)$ can also be thought of as a unit vector on the Fermi surface, since above in Eq. (C3) we only used the stationary phase condition where the momentum is aligned with the velocity.

Neglecting third-order terms in the difference variables the free particle action is

$$\begin{aligned} & \sum_\mu \int_0^t ds \left[p(s, \mu) v(s, \mu) - \frac{\mathbf{p}(s, \mu)^2}{2m} \right] \\ & \approx \int_0^t ds [P(s) \mathbf{n}(s) \cdot \Delta \dot{\mathbf{r}}(s) + \Delta p(s) V(s)] \\ & - \int_0^t ds \frac{P(s) \Delta p(s)}{m} \\ & + \mathbf{k} \cdot \left[[\mathbf{R}(t) - \mathbf{R}(0)] - \int_0^t ds V(s) \mathbf{n}(s) \right], \quad (\text{C5}) \end{aligned}$$

where \mathbf{k} is a Lagrange multiplier vector that has been introduced in order to deal with the boundary condition in the transition from \mathbf{R} to $\dot{\mathbf{R}}$. $(1/\hbar)|\mathbf{k}|V\tau_{\text{tr}}$ is a small number, when the feeding-in and the pickup are not local in space. As an illustration assume that the initial and final positions are distributed according to isotropic Gaussians centered around \mathbf{R}_i^0 and \mathbf{R}_f^0 , respectively. The sum of the variances is denoted by $(\hbar/\Delta k)^2$. The term $\mathbf{k} \cdot (\mathbf{R}_f - \mathbf{R}_i)$ in the action is then replaced by

$$\mathbf{k} \cdot (\mathbf{R}_f^0 - \mathbf{R}_i^0) + \frac{i\hbar}{2} \frac{k^2}{\Delta k^2} \quad (\text{C6})$$

after the integration with respect to \mathbf{R}_i and \mathbf{R}_f has been performed. The typical size of \mathbf{k} is now given by the smallest of the numbers $\hbar/(\mathbf{R}_f - \mathbf{R}_i)$ and $\Delta\lambda$. The last number is in the semiclassical limit small. In the case of a periodic driving \mathbf{k} is the wave vector of the external field. Anyway, in the semiclassical limit we will neglect the contribution from \mathbf{k} .

1. Interaction part: General formulation

We first Taylor expand the interaction part of the action S_{int} , the scalar part of Eqs. (56) and (57), to second order in the difference variable $\Delta\mathbf{r}$. Furthermore, we use a time-local approximation. That is, we approximate the quadratic terms $\Delta\mathbf{r}(s)\Delta\mathbf{r}(s')$ as $\Delta\mathbf{r}(s)^2$. Using this,

$$S_{\text{int}} \approx \frac{i}{2} \int_0^t ds \Delta\mathbf{r}(s) \cdot \mathbf{F}[\mathbf{R}, s] \cdot \Delta\mathbf{r}(s) - \int_0^t ds \Delta\mathbf{r}(s) \cdot \mathbf{h}[\mathbf{R}, P, s], \quad (\text{C7})$$

where

$$\mathbf{h}(\mathbf{R}, P, s) = h(V(s), P, s) \mathbf{n}(s), \quad (\text{C8})$$

$$\mathbf{F}(\mathbf{R}, s)_{\alpha\beta} = (\delta_{\alpha\beta} - \mathbf{n}_\alpha \mathbf{n}_\beta) F_{\text{T}}[V(s)] + \mathbf{n}_\alpha \mathbf{n}_\beta F_{\text{L}}[V(s)], \quad (\text{C9})$$

with

$$\begin{aligned} h(V(s), P, s) &= \int_0^t ds'' \frac{\partial D_{00}}{\partial |\mathbf{R}|} (V(s)(s-s''), s-s'') \\ & \times \sum_{\mu''} \left[\frac{1}{2} - w \left(P(s''^\dagger) + (-1)^{\mu''} \frac{\Delta p(s''^\dagger)}{2} \right) \right], \quad (\text{C10}) \end{aligned}$$

$$F_{\text{L}}[V(s)] = \int_0^t ds' \frac{\partial^2 C_{00}}{\partial |\mathbf{R}|^2} (V(s)|s'-s|, s'-s) \quad (\text{C11})$$

$$F_{\text{T}}[V(s)] = \int_0^t ds' \frac{1}{V(s)|s'-s|} \frac{\partial C_{00}}{\partial |\mathbf{R}|} (V(s)|s'-s|, s'-s). \quad (\text{C12})$$

As a further simplification we have above also approximated the average path by a straight line inside the kernels. In fact we will below treat both F_{T} and F_{L} as constants.

In this approximation the action separates into a longitudinal and a transverse part. We define

$$\Delta\mathbf{r}(s) = \Delta r_{\parallel}(s) \mathbf{n}(s) + \Delta r_{\perp}(s) \hat{\mathbf{z}} \times \mathbf{n}(s). \quad (\text{C13})$$

In the Wigner variable basis of the propagator we use a polar coordinate representation of the initial and final momenta:

$$\mathbf{P}_i = P_i \begin{pmatrix} \cos \phi_i \\ \sin \phi_i \end{pmatrix}, \quad (\text{C14})$$

$$\mathbf{P}_f = P_f \begin{pmatrix} \cos \phi_f \\ \sin \phi_f \end{pmatrix}. \quad (\text{C15})$$

In the full propagator we now split out all the factors related to the transverse direction into the so-called transverse propagator

$$\begin{aligned} J_{\text{T}} &= \int \mathcal{D}V_\phi \int \mathcal{D}\Delta r_{\perp} \frac{2\pi\hbar}{P_f} \delta \left[\phi_f - \frac{V_\phi(t)}{V(t)} \right] \frac{2\pi\hbar}{P_i} \\ & \times \delta \left[\phi_i - \frac{V_\phi(0)}{V(0)} \right] \exp \left(-\frac{1}{\hbar} \int_0^t ds \left[\frac{1}{2} F_{\text{T}} \Delta r_{\perp}(s)^2 \right. \right. \\ & \left. \left. + i \Delta r_{\perp}(s) P(s) \frac{d}{ds} \frac{V_\phi(s)}{V(s)} \right] \right). \quad (\text{C16}) \end{aligned}$$

In terms of J_{T} the full propagator is, after integrating out $\Delta\mathbf{r}(0)$ and $\Delta\mathbf{r}(t)$,

$$\begin{aligned}
J = & \int \mathcal{D}P \int \mathcal{D}\Delta p \int \mathcal{D}V \int \mathcal{D}V \Delta r_{||} 2\pi\hbar \\
& \times \delta[P_f - P(t)] 2\pi\hbar \delta[P_i - P(0)] J_T \\
& \times \exp \left\{ \frac{i}{\hbar} \int_0^t ds \left[\Delta p(s) V(s) - \frac{1}{m} P(s) \Delta p(s) \right] \right. \\
& - \frac{1}{\hbar} \int_0^t ds \left[\frac{1}{2} \Delta r_{||}(s)^2 F_L + i \Delta r_{||}(s) \right. \\
& \left. \left. \times \left(\frac{d}{ds} P(s) + h[V(s), P, s] \right) \right] \right\}. \quad (C17)
\end{aligned}$$

To lowest order the transverse propagator is independent of the dynamics of the longitudinal variables, so we will treat J_T as a constant and absorb it in the overall normalization of the path integral. The study of the remaining longitudinal propagator is the topic of the next subsection.

2. Interaction part: The longitudinal contribution

In the propagator, Eq. (C17), we will neglect the dependence of h upon Δp . $V(s)$ thus equals $m^{-1}P(s)$. The propagation described by Eq. (C17) is then equivalent to the quantum propagation problem

$$-\hbar \partial_s = \frac{1}{2} \left(\frac{\hbar}{i} \frac{\partial}{\partial P} \right)^2 F_L(P) - \hbar \frac{\partial}{\partial P} h(P). \quad (C18)$$

We neglect the momentum dependence of F_L . In equilibrium it is

$$h(P) \approx \alpha \frac{1}{2} \tanh \left[\frac{v_F}{2k_B T} (P - p_F) \right], \quad (C19)$$

where the constant α in the high-temperature limit is related to the fluctuations through

$$\frac{\hbar F_L}{\alpha} = \frac{k_B T}{v_F}. \quad (C20)$$

The last term in Eq. (C18) is gauged away by the transformation $\phi(P) (-\partial f_F / \partial p)^{-1/2}$ and the propagation problem, Eq. (C18), is equivalent to

$$\begin{aligned}
-2 \frac{\hbar F_L}{\alpha^2} \partial_s \phi = & - \frac{\hbar^2 F_L^2}{\alpha^2} \frac{\partial^2 \phi}{\partial P^2} \\
& + \left[- \frac{\hbar F_L}{\alpha} \frac{\partial}{\partial P} \tanh \left(\frac{1}{2} \frac{v_F}{k_B T} (P - p_F) \right) \right. \\
& \left. + \tanh^2 \left(\frac{1}{2} \frac{v_F}{k_B T} (P - p_F) \right) \right] \phi. \quad (C21)
\end{aligned}$$

The ground state of the quantum problem, Eq. (C21), is the derivative of the Fermi distribution function and the ground-state energy is zero. The typical decay time of the excited states is given by $\hbar F_L / \alpha^2$. This can be identified as the correlation time scale of $\Delta r_{||}$ or P with themselves. It is also the time scale it takes before the boundary conditions cannot be felt anymore. For the Coulomb interaction this time scale is approximately $\hbar (k_B T)^{-1} \hbar g_{1P_F}^2 (v_{FP_F})^{-1}$. The ground-state wave function of the original quantum propagation problem, Eq. (C18), is proportional to $\sqrt{-\partial f_F / \partial p}$. This implies that the distribution of the longitudinal momentum or energy is given by the derivative of the Fermi distribution function, while the average of the width squared is

$$\begin{aligned}
& \left\{ \int_{-\infty}^{\infty} dp \left(- \frac{\partial f_F}{\partial p} \right) \right\}^{-1} \int_{-\infty}^{\infty} dp \left(\frac{\hbar}{2} \frac{\partial^2 f_F}{\partial p^2} \right)^2 \left(- \frac{\partial f_F}{\partial p} \right)^{-1} \\
& = \frac{1}{3} \frac{\hbar^2 v_F^2}{(k_B T)^2}. \quad (C22)
\end{aligned}$$

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