

THE VORTEX PICTURE OF THE QUANTUM HALL EFFECT

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In this contribution we discuss the vortex picture of the fractional quantum Hall effect. Provided the vortices remained pinned in the presence of a transport current we show that this vortex picture implies the existence of plateaus. Finally we give an argument which elucidates why the ground state at $\nu \neq 1/m$ is an inhomogeneous vortex state rather than a homogeneous state.

1. Introduction

When the Hall-resistance, V_H/I , of a 2-dimensional electron gas is depicted as a function of the magnetic field, B , the experimental curves display plateaus around filling factors $\nu = p/q$, where p and q are small integers, and q is odd. The plateaus are positioned with an extraordinary accuracy at values $V_H/I = (q/p)h/e^2$, showing that we are dealing with a macroscopic quantum effect. For $q = 1$ we speak about the integer quantum Hall effect (IQHE) and when $q = 3, 5, 7, \dots$ the phenomenon is called the fractional quantum Hall effect (FQHE).

The plateau is the hall mark of the quantum Hall effect, and the basic theoretical challenge is to explain the formation of plateaus and in particular to explain the extreme accuracy of the V_H/I -quantization. In the present paper we show that together with a pinning assumption, the vortex picture of the FQHE leads to the formation of plateaus, and we discuss the possibility that the vortex picture may apply also to the IQHE.

In section 2 we review the essential features of the vortex picture of the FQHE [1,2,3,4]. The nature of the vortex coordinate is discussed in section 3. Then, in section 4, we discuss why the ground state has to become inhomogeneous when the filling factor differs slightly from $1/m$. Finally, in section 5, we show how the vortex picture implies the existence of plateaus. The argument presented in section 5 is consistent with and supplementary to the argument given in [3], but it is simpler, because an explicit discussion of the pinning forces is avoided.

2. The vortex picture

The vortex picture of the FQHE [2,3,4] that has emerged from Laughlin's theory [1] of the incompressible electron liquid is characterized by the following features: at the filling factor $\nu = p/q$ the ground state is a homogeneous liquid. However, if by a small change of the magnetic field B the situation $\nu \neq p/q$ is produced, the ground state becomes inhomogeneous by containing topological defects in the form of vortices or antivortices. In the region between the topological defects the electron density varies with B as

$$n(B) = \frac{B}{B_\nu} n_0, \tag{1}$$

where $B_\nu = (n_0 h)/(e\nu)$ is the magnetic field corresponding to $\nu = p/q$, and n_0 is the average electron density i.e. the density of the homogeneous mid-plateau state at $B = B_\nu$. We emphasize that for any given filling factor $\nu = p/q$, the relationship (1) will be fulfilled only in an interval around p/q (finite plateau width). According to (1) the level of the electron liquid rises in the region between the topological defects as the magnetic field is increased, and it is lowered when the magnetic field is decreased.

Within the Laughlin theory, treating the case $\nu = 1/m$, the shapes and velocity fields of the topological defects follow from his quasi-hole and quasi-electron wave functions. In his theory the presence of a vortex at the position z_0 is described by

$$\Psi_h = \exp\left(-\sum_{n=1}^N \frac{|z_n|^2}{4l^2}\right) \prod_{i=1}^N (z_i - z_0) \prod_{k < j}^N (z_k - z_j)^m, \tag{2}$$

where $z = x - iy$. The state (2) has a reduced electron density and a circulating current around z_0 , so the image of this state is a vortex of the bathtub kind.

Let us consider the situation where the magnetic field has been increased from B_ν to B . In this case the ground state contains $Nm(B/B_\nu - 1)$ vortices, where N is the number of electrons. Within each vortex region there is a charge deficit of $1/m$ of an electron charge. The charge removed from the vortex regions will appear in the region between the vortices. Thus by charge conservation equation (1) will be fulfilled.

In the presence of disorder, say the fluctuating potential from ionized donor atoms, the vortices will be localized in local energy minima.

The relationship (1) is valid in the bulk of the sample. The conditions near the edges, which is of particular importance for narrow channels, requires a special treatment, but we shall not discuss the role of edge states in the present context.

3. The nature of the vortex coordinate

A priori one might think of the vortex coordinate, z_0 , in equation (2) as either (a) a quantum mechanical position variable, or (b) a parameter. In the first case we must treat Ψ_h as a function of $N + 1$ quantum mechanical position variables $z_0, z_1, z_2, \dots, z_N$, and consequently the electron density is given as $n(z_1) \propto \int \dots \int dz_0 dz_2 \dots dz_N \Psi_h^* \Psi_h$, i.e. the electron density is homogeneous. In the second case the electron density is $n(z_1) \propto \int \dots \int dz_2 \dots dz_N \Psi_h^* \Psi_h$ displaying a reduced density around z_0 . In order to investigate which of the two possibilities that is appropriate we consider the many particle state, Φ , that results when, from a filled lowest Landau level containing $N + 1$ electrons, we remove an electron from the state $\Psi_{0,0}$. The state Φ is described by the Slater determinant

$$\Phi = \frac{1}{\sqrt{N!}} \begin{vmatrix} \Psi_{0,1}(z_1) & \Psi_{0,1}(z_2) & \dots & \Psi_{0,1}(z_N) \\ \Psi_{0,2}(z_1) & \Psi_{0,2}(z_2) & \dots & \Psi_{0,2}(z_N) \\ \vdots & \vdots & \ddots & \vdots \\ \Psi_{0,N}(z_1) & \dots & \dots & \Psi_{0,N}(z_N) \end{vmatrix} \quad (3)$$

where

$$\Psi_{0,p} = (2\pi l^2 2^p p!)^{-\frac{1}{2}} \left(\frac{z}{l}\right)^p \exp\left(-\frac{|z|^2}{4l^2}\right) \quad (4)$$

are the single particle states of the lowest Landau level. The state Φ is a vortex state [5] with a circulating current $I = \mu_B/2\pi l^2$ around the vortex centre and a density $n(|z|) = (1 - \exp(-|z|^2/2l^2))/(2\pi l^2)$. Actually Φ is identical with the Laughlin vortex wave function given in (2) for $m = 1$ and $z_0 = 0$. Since the state Φ is stationary in the lowest Landau level approximation of the Laughlin theory, the centre-coordinate, z_0 , of the vortex will maintain its sharp value in the course of time. In contradistinction any proper quantum mechanical position variable cannot maintain a sharp value for any span of time. Therefore we conclude that z_0 must be treated as a parameter.

4. Why not a homogeneous ground state at $\nu \neq 1/m$?

In this section we elucidate what prevents the existence of a homogeneous state at $\nu \neq 1/m$, with an energy lower than the vortex state.

Considering the ground state wave function

$$\Psi = \Psi(\mathbf{r}_1, \dots, \mathbf{r}_N) = f \exp(i\phi) \quad (5)$$

we initially imagine that all the electrons, except electron 1, are kept at arbitrary fixed positions. The velocity field of electron 1 is then given as

$$\mathbf{v}_1 = \frac{\hbar}{M} \nabla_1 \phi + \frac{e}{M} \mathbf{A}_1, \quad (6)$$

where $\mathbf{A}_1 = \frac{1}{2} B (-y_1, x_1, 0)$. Now consider the singular gauge transformation [6]

$$\mathbf{a}_j = \frac{m\Phi_0}{2\pi} \sum_{i \neq j}^N \nabla \text{Im} \ln(z_j - z_i) \quad (7)$$

which attaches an infinitely thin flux tube with m flux quanta to each particle, but which does not give rise to any magnetic forces. This gauge transformation [7] changes Laughlin's ground state wave function into $\Psi'_L = \prod_{i>j}^N |z_j - z_i|^m \exp(-1/(4l^2) \sum |z_k|^2)$, showing that the $\nabla_1 \phi$ term in (6) vanishes in the singular gauge. In this gauge we therefore obtain

$$\mathbf{v}_1 = \frac{e}{M} \mathbf{A}'_1 \quad (8)$$

where $\mathbf{A}'_1 = \mathbf{A}_1 + \mathbf{a}_1$.

Considering first the midplateau state at $\nu = 1/m$, we calculate the circulation of \mathbf{v}_1 along an arbitrary closed curve. Since $\oint \mathbf{A} \cdot d\mathbf{s}_1 = B\sigma$, where σ is the enclosed area, we obtain

$$\oint \mathbf{v}_1 \cdot d\mathbf{s}_1 = \frac{e}{M} \oint \mathbf{A}'_1 \cdot d\mathbf{s}_1 = \frac{e}{M} (B\sigma - N_\sigma m \Phi_0), \quad (9)$$

where N_σ is the number of electrons inside the closed curve. Releasing the fixed electrons and averaging over their positions we obtain

$$\langle \oint \mathbf{v}_1 \cdot d\mathbf{s}_1 \rangle = \frac{e}{M} (B\sigma - n_0 \sigma m \Phi_0) = 0, \quad (10)$$

where we have used that $\langle N_\sigma \rangle = \sigma n_0$. In a mean field description at $\nu = 1/m$ the average contribution to the velocity field from the statistical vector potential \mathbf{a}_1 exactly cancels the contribution from the physical vector potential \mathbf{A}_1 .

Then let us consider what would happen if the density remained homogeneous after the magnetic field has been changed into $B = B_0 + \Delta B$. Taking the line integral along a circle of radius R we find from (8) that

$$\langle \oint \mathbf{v}_1 \cdot d\mathbf{s}_1 \rangle = \frac{e}{M} \pi R^2 \Delta B \Rightarrow v = \frac{1}{2} \frac{e}{M} \Delta B R. \quad (11)$$

At $\nu \neq 1/m$ the contributions from the statistical and the physical vector potential no longer cancels. As a consequence the entire electron liquid will rotate with a velocity field that increases linearly with R . This absurd conclusion of course arises because we assumed that the electron density remained homogeneous. In reality the electron liquid solves its rotational problem, not by rotating as a whole, but by the setting up of many small rotations in the form of vortices or antivortices. However in the region between the vortices we still have $\langle \mathbf{A}'_1 \rangle = 0$.

5. Plateau formation

We consider the situation where a current is passed through a rectangular sample with ν slightly less than $1/m$, so that the electron liquid contains a collection of vortices. First imagine the idealized case where there is no disorder to pin the vortices. In this case the entire vortex pattern flows along with the liquid. At an arbitrary fixed position in the sample the electron density will therefore fluctuate between n , when a vortex free region is passing, and a reduced density each time a vortex passes our point of observation. The time average of the electron density is therefore n_0 . But since force balance requires that the electron liquid, with its vortex pattern, flows along with the velocity $v = E/B$ the time average of the current density is $j = en_0 E/B$,

which is the free electron result, implying the absence of plateaus.

Then consider a real sample where the vortices are expected to remain pinned in the limit of low temperature and small currents. In this case the inter-vortex region is stationary and it will therefore be possible to establish a stationary path (from one side of the sample to the other) which is everywhere well away from any of the pinned vortices [5]. Thus along this path we have

$$n = n_0 \frac{B}{B_\nu} \Rightarrow \nu = \frac{1}{m} \Rightarrow \mathbf{j} = \frac{1}{m} \frac{e^2}{h} \frac{\mathbf{B}}{B} \times \mathbf{E} \quad (12)$$

Therefore, by integration along this path we obtain the same Hall response,

$$V_H = m \frac{h}{e^2} I, \quad (13)$$

as in the mid-plateau state. But this is tantamount to the formation of a plateau. A more detailed discussion, in particular with respect to possible minute deviations from (13), is given in [5].

Also in the general fractional case $\nu = p/q$ the electron ground state is supposed to form an incompressible liquid with a homogeneous density. As the filling factor is made slightly different from p/q topological defects arise in the liquid. However, the essential thing is that (1) is fulfilled in the intervortex region, so that the explanation of plateau formation around p/q is the same as for the $\nu = 1/m$ -case.

The p/q -states have been described in terms of the hierarchy theory [8,9,10], according to which the quasi-particles, i.e. the vortices, are supposed to condense into a Laughlin-type correlated state. Recently, however, the appropriateness of the hierarchy theory has been questioned [11]. Although the hierarchical theory offers a possible classification of the FQHE states, the theory has some conceptual difficulties. As more and more vortices are created in the liquid the correlations characteristic of the $\nu = 1/m$ -state are gradually weakened. However, it takes a large number of vortices to form a Laughlin-type condensate, in fact so large a number that the average distance between the vortices is comparable to the size of a vortex. This suggests that the vortices may have stopped to be well defined physical objects (or to exist at all) before there are enough of them to form the new condensate of the daughter state. Also, the feature that the vortex centre coordinate is a parameter rather than a quantum mechanical position variable, makes it unclear to which extend the vortices will be able to imitate the $\nu = 1/m$ -condensation. In our view, it is still an open question how the incompressibility of the $\nu = p/q$ -states is best accounted for.

6. Discussion

At present, the formation of plateaus in the IQHE and the formation of plateaus in the FQHE are generally considered as rather distinct phenomena. Whereas

plateau formation in the IQHE is considered to be a single-particle localization phenomenon, the plateaus in the FQHE simply would not be there if it were not for the electron-electron interaction. Nevertheless, the possibility of establishing a unified theory of plateau formation has been discussed in a few papers [11,12,13,3]. In this context it is appropriate to ask the following question concerning the plateau formation in the IQHE: is equation (1) fulfilled in an interconnected region with such a topology that the formation of plateaus around integer filling factors can be explained by the same argument, given in section 5, as the formation of plateaus around filling factors $\nu = 1/m$? This will be the case if the ground states around integer filling factors are vortex states. But it can also be the case if there exists a phase separation into incompressible and compressible regions [13,14,15]. In the latter case (12) will be fulfilled in the incompressible regions whereas the current density is zero in the compressible regions. Both possibilities implies that electron-electron interaction plays a role also in the IQHE.

To illustrate the meaning of vortex states around integer filling factors we have considered the case where ν is slightly less than one [5]. In this case the velocity field and the density deviation of the vortices, distributed around in the liquid, will be rather much like that displayed by the state (3).

In our view all plateau formation results from the fulfillment of (1) in an interconnected region with a suitable topology, and we find it most likely that fulfillment of (1) in the FQHE as well as in the IQHE is due to the existence of vortices.

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