2D and 3D flow simulations in porous material

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Chapter 1

Introduction

Porous materials are widely used in filtering processes in the industries to filter solutions of different species in a fluid. In order to make an efficient filter from a porous material, it is necessary to understand how a fluid flows through a porous filter. The goal of this report is then to get an understanding of the flow in a porous material.

A theoretical model of a pipe with a plug of a porous material will be used to describe the flow in a porous material and it will be investigated how the pressure develops through a channel with porous walls.

On this theoretical basis 2D and 3D simulations are made of two different filters: a filter with 7 channels and a filter with 19 channels. In the 2D simulations the flux from the channels is investigated for different pressures and it is furthermore studied how the flux from the channels into the porous material, called the substrate, depends on the permeability, i.e. how porous the substrate is.

In the 3D simulations the pressure distribution in the channels along the length of the filter is studied for different permeabilities in the substrate. The flux from the channels into the substrate is also investigated for different permeabilities at specified sections along the length of the filter. The total flux from the filter is compared for the two filters at the different permeabilities in order to see the effect of increasing the number of channels in the filter.
CHAPTER 1. INTRODUCTION
Chapter 2

Geometry

The 2D model of the filter with seven channels, called a 7-channel filter, is seen in figure 2.1. The seven channels are placed such that one channel, called the inner channel, is placed in the center of the filter and the other six channels (the peripheral channels) are placed in a distance $R_2 = 0.5$ m from the center. The six peripheral channels are placed symmetrically with $\theta = 60^\circ$ between them. The filter has a radius of $R_1 = 1$ m and the channels have a radius of $R_c = 0.1$ m. The porous material between the channels is called the substrate.

![Figure 2.1: The dimension of the 7-channel filter, with one channel in the center and the six others placed $R_2 = 0.5$ m from the center with $\theta = 60^\circ$ between each other. The filter has a radius of $R_1 = 1$ m and the channels a radius of $R_c = 0.1$ m.](image)

The 3D model is shown from the side in figure 2.2. This model is an extruded version of the 2D model shown in figure 2.1, meaning that the 3D model looks like the 2D model, when seen from the front. The length of the filter is $L = 5$ m, and the pressure at the inlet is $p_{in}$, at the outlet it is $p_{out}$ and $p_s$ at the surface of the substrate.
Figure 2.2: A view from the side of the 3D filter. The model is an extruded version of the 2D model shown in figure 2.1.
Chapter 3

Theory

The theoretical basis of the report is presented in this chapter. Sections 3.2 and 3.3 are written with inspiration from [2].

Darcy’s linear law for the velocity in a porous material is found by studying the flow through a porous plug in a pipe. An external pressure drop is put on the pipe and it is assumed that the flow in the pipe is laminar, whereby it can be described as a Poiseuille flow.

The pressure in a pipe with porous walls is studied in order to see how the pressure develops through the pipe as fluid flows out through the walls. Here there is again assumed to be laminar flow in the pipe and it is furthermore assumed that Darcy’s law is valid in the porous material.

In the last section both the pipe and the walls are modeled as porous materials. Again it is studied how the pressure develops through the pipe. Darcy’s law is assumed to be valid in both the pipe and in the walls.

3.1 Fundamental equations

The Navier-Stokes equation is the fundamental partial differential equation describing the motion of fluids by conservation of momentum. For an incompressible, Newtonian fluid it can be expressed as

\[ \rho \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p + \eta \nabla^2 \mathbf{v} + \rho \mathbf{g} \] (3.1)

The left-hand side can be interpreted as the inertia force densities, while the right-hand side is the applied force densities. In this report, the gravitational term \( \rho g \) is neglected.

The Reynolds number is often used in fluid mechanics, since it describes relative importance of inertia to viscous forces. Conventionally it yields

\[ \text{Re} = \frac{\text{Inertia}}{\text{Viscosity}} = \frac{\rho U L}{\eta} \] (3.2)

where \( \rho \) is the density, \( U \) is the characteristic velocity, \( L \) is the characteristic length scale and \( \eta \) is the dynamic viscosity. Throughout this report it is assumed that the Reynolds number is very low. This means that the viscous forces are dominating in the flow and
thus the left-hand side of the Navier-Stokes equation can be neglected. Equation 3.1 then reduces to

$$0 = -\nabla p + \eta \nabla^2 \mathbf{v} \quad (3.3)$$

which is the Stokes equation.

### 3.2 Pipe with a plug

A simple flow problem is used in order to find an analytical solution to the velocity in a porous media. The flow problem consists of an infinitely long, translation-invariant pipe with a plug of some porous material, see figure 3.1. The radius of the pipe is $R$ and the width of the porous plug is $w$. An external pressure differences is put on the pipe. The flow in the pipe is assumed to be laminar and the flow can thus be described as a Poiseuille flow. The flow in the porous plug is a Darcy flow and in order to describe this, equation 3.3 is extended with a dissipative force against the flow

$$0 = -\nabla p + \eta \nabla^2 \mathbf{v} - \alpha \mathbf{v} \quad (3.4)$$

The dissipative term originates in the porous material and $\alpha$ is given as, $\alpha = \frac{\eta}{k}$ where $k$ is the permeability in the porous material. Outside of the porous material, $\alpha = 0$ since $k \to \infty$.

![Figure 3.1: Plug of porous material in a pipe with an external pressure difference.](image)

In the following the focus will be on the flow through the porous plug. The pressure gradient is assumed to be constant in the porous material, i.e. $\nabla p = \frac{\Delta p}{w}$, where $w$ is the width of the porous material. Herby equation 3.4 becomes

$$0 = -\frac{\Delta p}{w} + \eta \nabla^2 \mathbf{v} - \alpha \mathbf{v} \quad (3.5)$$

By use of cylindrical coordinates equation 3.5 becomes

$$0 = -\frac{\Delta p}{w} + \eta (\partial_r^2 v_x + \frac{1}{r} \partial_r v_x) - \alpha v_x \quad \Rightarrow$$

$$\frac{1}{r} \partial_r v_x + \partial_r^2 v_x = -\frac{1}{\eta} \frac{\Delta p}{w} + \frac{1}{k} v_x \quad (3.6)$$

The following parameters are used to non-dimensionalize equation 3.6

$$l = \sqrt{k}, \quad v_0 = \frac{k}{\eta} \frac{\Delta p}{w}, \quad s = \frac{r}{l}, \quad \ddot{s} = \frac{R}{l}, \quad u(s) = \frac{v_x}{v_0}$$
3.2. PIPE WITH A PLUG

By insertion this leads to

\[
\frac{1}{s^2} u'(s) \frac{v_0}{L} + u''(s) \frac{v_0}{L^2} = -\frac{1}{\eta \omega} \Delta p + \frac{1}{k} u(s) v_0 \quad \Rightarrow
\]

\[
\frac{1}{s^2} u'(s) \frac{k \Delta p}{\eta \omega} + u''(s) \frac{k \Delta p}{\eta \omega L^2} = -\frac{1}{\eta \omega} \Delta p + \frac{1}{k} u(s) \frac{k \Delta p}{\eta \omega} \quad \Rightarrow
\]

\[
u''(s) \frac{k}{L^2} + \frac{k}{s^2} u'(s) - u(s) + 1 = 0 \quad \Rightarrow
\]

\[
u''(s) + \frac{1}{s} u'(s) - u(s) + 1 = 0 \quad (3.7)
\]

The solution to this 2. order inhomogeneous differential equation is given as

\[
u(s) = u_h(s) + u_p(s) \quad (3.8)
\]

A particular solution is easily seen to be \(u_p(s) = 1\). The homogeneous part of the differential equation can be described as a modified Bessel equation. The modified Bessel equation on general form is given as

\[
x^2 y'' + xy' - (x^2 + n^2)y = 0, \quad n \geq 0 \quad (3.9)
\]

In this case \(n = 0\), i.e. the homogenous part of the equation is a modified bessel function of zeroth order. The solution can then be found as

\[
u_h(s) = C_2 I_0(s) + C_1 K_0(s) \quad (3.10)
\]

The full solution is given as the particular solution plus the homogenous solution, i.e.

\[
u(s) = C_2 I_0(s) + C_1 K_0(s) + 1 \quad (3.11)
\]

The boundary conditions to this problem are no-slip at the walls, \(u(\tilde{s}) = 0\), and a symmetry condition, \(\partial_r u(r) = 0\) \(|r=0|\). As \(\partial_r K_0(s)\) diverges for \(r \to 0\), it is given that \(C_1 = 0\). This reduces the solution to

\[
u(s) = C_2 I_0(s) + 1 \quad (3.12)
\]

Using the no-slip condition, \(C_2\) can be found

\[
u(\tilde{s}) = C_2 I_0(\tilde{s}) + 1 = 0 \quad \Rightarrow \quad C_2 = \frac{1}{-I_0(\tilde{s})} \quad (3.13)
\]

The solution for \(u(s)\) then becomes

\[
u(s) = 1 - \frac{I_0(s)}{I_0(\tilde{s})} \quad (3.14)
\]

The solution is investigated in the two limits, \(\tilde{s} \ll 1\) and \(\tilde{s} \gg 1\). In the case of \(\tilde{s} \ll 1\) i.e. \(R \ll \sqrt{k}\), \(I_0(s)\) is found as

\[
I_0(s) = \frac{1}{2} \sum_{m=0}^{\infty} \frac{(\frac{1}{2} s^2)^m}{m! \Gamma(m+1)} \quad (3.15)
\]
where $\Gamma(m+1) = m!$, which then leads to

$$I_0(s) = \sum_{m=0}^{\infty} \frac{(\frac{1}{2}s)^{2m}}{(m!)^2} \Rightarrow$$

$$I_0(s) = \sum_{m=0}^{\infty} \frac{1}{(2m!)^2} s^{2m} \Rightarrow$$

$$I_0(s) \approx 1 + \frac{1}{2} \frac{s^2}{\pi}$$

(3.16)

This expression for $I_0(s)$ and $\tilde{I}_0(s)$ is inserted into equation 3.14

$$u(s) = 1 - \frac{1 + \frac{1}{2} \frac{s^2}{\pi}}{1 + \frac{1}{2} \frac{s^2}{\pi}} = 1 - (1 + \frac{1}{2} \frac{s^2}{\pi})(1 + \frac{1}{2} \frac{s^2}{\pi})^{-1}$$

(3.17)

In order to simplify this equation further, a Taylor expansion of the last part is performed, by expanding the term $(1 + x)^{-1}$, where $x = \frac{1}{2} \frac{s^2}{\pi}$

$$\sum_{m=0}^{\infty} \frac{f^m(a)}{m!} (x - a)^m \approx 1 - (1 + a)^{-2}(x + a) \approx 1 - x$$

(3.18)

Hereby the following simple expression can be found for $u(s)$

$$u(s) = 1 - (1 + \frac{1}{4} s^2)(1 - \frac{1}{4} s^2)$$

$$u(s) = \frac{1}{4} s^2 - \frac{1}{4} s^2$$

$$u(s) = \frac{1}{4} (s^2 - s^2)$$

(3.19)

Inserting lengths and velocity into this equation, the solution to the velocity is

$$v_x(r) = \frac{\Delta p}{4 \eta w} (R^2 - r^2), \quad \text{for } R \ll \sqrt{k}$$

(3.20)

Equation 3.20 is recognized as the Poiseuille flow solution for laminar flow in a pipe. This means that the flow in the porous material will behave like a normal pipe flow, which is expected because when the permeability, $k$, is large, the material will not give any noticeable resistance to the flow. The dissipative term added in equation 3.4 could then be neglected in this case, because $\alpha = \frac{\eta}{k} \to 0$ for $k \to \infty$.

In the other limit $\hat{s} \gg 1$ and $R \gg \sqrt{k}$. In this case $I_0(s)$ has an asymptotic behavior

$$I_0(s) = \frac{1}{\sqrt{2\pi} s}$$

(3.21)
This is inserted into equation 3.14

\[ u(s) = 1 - \frac{1}{\sqrt{2\pi s}} e^{-\frac{s^2}{2}} \]

\[ u(s) = 1 - \sqrt{s} e^{-\left(\tilde{s}^2 - s^2\right)} \tag{3.22} \]

It has been chosen to use \( \tilde{s} = 10^4 \), because this magnitude agrees well with the limit, \( \tilde{s} \gg 1 \). This leads to the following

\[ u(s) = \begin{cases} 
1 & \text{for } \frac{\tilde{s}}{s} - 1 > 10^{-3} \\
1 - e^{-\left(\tilde{s}^2 - s^2\right)} & \text{for } \frac{\tilde{s}}{s} - 1 < 10^{-3}
\end{cases} \]

With the chosen \( \tilde{s} \), \( u(s) \) only deviates 0.1% from unity near the wall. Because of such a small deviation, it is a good approximation to assumed that the velocity profile is constant

\[ u(s) \equiv 1 \tag{3.23} \]

Inserting the non-dimensional velocity into equation 3.23 yields

\[ v_x(r) = v_0 = \frac{k \Delta p}{\eta w}, \quad R \gg \sqrt{k} \tag{3.24} \]

It is then seen that the velocity in the porous material is constant and equation 3.24 is recognized as Darcy’s linear law. From this it can be concluded that the velocity in the porous material can be approximated by Darcy’s linear law.

### 3.3 Channel with porous walls

In this section a flow through a pipe with porous walls will be investigated, see figure 3.2. The figure is borrowed from [2]. The flow is assumed to be laminar and Darcy’s law

\[ Q_{\text{trans}}(x) = p_{\text{perm}}, \quad \Delta p = \frac{k \Delta p}{\eta w} \]

\[ Q_{\text{pipe}}(x) = Q_{\text{pipe}}(x + dx) \]

\[ P_{\text{in}} \]

\[ P_{\text{out}} \]

\[ x = 0 \]

\[ x = L \]

\[ dx \]

\[ w \]

\[ R \]

\[ Q_{\text{pipe}}(x) \]

\[ Q_{\text{pipe}}(x + dx) \]

\[ P_{\text{out}} \]

\[ P_{\text{in}} \]

\[ x = 0 \]

\[ x = L \]

\[ dx \]

\[ w \]

\[ R \]

\[ Q_{\text{pipe}}(x) \]

\[ Q_{\text{pipe}}(x + dx) \]

\[ P_{\text{out}} \]

\[ P_{\text{in}} \]

\[ x = 0 \]

\[ x = L \]

\[ dx \]

\[ w \]

\[ R \]

\[ Q_{\text{pipe}}(x) \]

\[ Q_{\text{pipe}}(x + dx) \]

\[ P_{\text{out}} \]

\[ P_{\text{in}} \]

\[ x = 0 \]

\[ x = L \]

\[ dx \]

\[ w \]

\[ R \]

\[ Q_{\text{pipe}}(x) \]

\[ Q_{\text{pipe}}(x + dx) \]

\[ P_{\text{out}} \]

\[ P_{\text{in}} \]

\[ x = 0 \]

\[ x = L \]

\[ dx \]

\[ w \]

\[ R \]

\[ Q_{\text{pipe}}(x) \]

\[ Q_{\text{pipe}}(x + dx) \]

\[ P_{\text{out}} \]

\[ P_{\text{in}} \]

\[ x = 0 \]

\[ x = L \]

\[ dx \]

\[ w \]

\[ R \]

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\[ Q_{\text{pipe}}(x + dx) \]

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\[ P_{\text{in}} \]

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\[ Q_{\text{pipe}}(x) \]

\[ Q_{\text{pipe}}(x + dx) \]

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\[ Q_{\text{pipe}}(x) \]

\[ Q_{\text{pipe}}(x + dx) \]

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\[ R \]

\[ Q_{\text{pipe}}(x) \]

\[ Q_{\text{pipe}}(x + dx) \]

\[ P_{\text{out}} \]

\[ P_{\text{in}} \]

\[ x = 0 \]

\[ x = L \]

\[ dx \]

\[ w \]

\[ R \]

\[ Q_{\text{pipe}}(x) \]

\[ Q_{\text{pipe}}(x + dx) \]

\[ P_{\text{out}} \]

\[ P_{\text{in}} \]

\[ x = 0 \]

\[ x = L \]

\[ dx \]

\[ w \]

\[ R \]

\[ Q_{\text{pipe}}(x) \]
is assumed to describe the flow in the porous material. In a section of the domain with
the length \(dx\), the mass conservation states that the correlation between the incoming
flow rate \(Q_p(x)\), the outgoing flow rate, \(Q_p(x + dx)\) and the flow rate through the porous
material, \(Q_{tm}(x)\), is given as

\[ Q_p(x) = Q_p(x + dx) + Q_{tm}(x) \tag{3.25} \]

The flow in the pipe is assumed to be Poiseuille flow and because of this the Hagen-
Poiseuille law can be used. The flow rate \(Q_p(x)\) in the circular pipe then becomes

\[ Q_p(x) = -\frac{1}{R_{hyd}} p'(x) \Rightarrow Q_p(x) = -\frac{\pi R^4}{8\eta} p'(x) \tag{3.26} \]

where the minus sign is due to the negative pressure gradient. The flow in the porous
material is described by the linear Darcy’s law, equation 3.24, and the velocity in the
porous material is therefore given as

\[ v(x) = -\frac{k}{\eta} p'(x) \tag{3.27} \]

where the minus sign once again is due to the negative pressure gradient. From this
expression it is possible to find the flow rate through the porous material. The pressure
difference across the porous material is given as \(\Delta p(x)\), so the pressure gradient across the
porous material can be written as \(p'(x) = -\frac{\Delta p(x)}{w}\), where \(w\) is the thickness of the wall.

\[ Q_{tm}(x) = -\frac{2\pi R k}{\eta} p'(x) dx \Rightarrow Q_{tm}(x) = \frac{2\pi R k}{\eta} \frac{\Delta p(x)}{w} dx \tag{3.28} \]

By inserting equations 3.26 and 3.28 into equation 3.25, the following expression is found

\[ \frac{p'(x + dx) - p'(x)}{dx} = \frac{16k}{R^3 w} \Delta p(x) \tag{3.29} \]

Introducing the relation \(\frac{p'(x + dx) - p'(x)}{dx} \approx p''(x)\), equation 3.29 becomes

\[ p''(x) = \frac{1}{\lambda^2} \Delta p(x) \tag{3.30} \]

where the constant \(\lambda\) is given as

\[ \lambda = \sqrt{\frac{R^3 w}{16k}} \tag{3.31} \]

The solution to equation 3.30 must be an exponential function. Since \(\Delta p(x) = p(x) - p_{perm}\),
the solution can be written on the form

\[ p(x) - p_{perm} = Ae^{x/\lambda} + Be^{-x/\lambda} \tag{3.32} \]
The boundary conditions to this problem are, \( p(0) = p_{in} \) and \( p(L) = p_{out} \), as seen on figure 3.2. By use of these boundary conditions the solution can now be written as

\[
p(x) = \frac{p_{out} - p_{perm} + (p_{perm} - p_{in})e^{-L/\lambda}}{e^{L/\lambda} - e^{-L/\lambda}}e^{x/\lambda} - \frac{p_{out} - p_{perm} + (p_{perm} - p_{in})e^{L/\lambda}}{e^{L/\lambda} - e^{-L/\lambda}}e^{-x/\lambda} + p_{perm} \tag{3.33}
\]

The solution to the pressure, \( p(x) \), can be rewritten by use of Taylor expansion and by setting \( p_{perm} = 0 \).

\[
p(x) = p_{in} + \frac{x}{L} \left[ p_{out} - p_{in} - \left( \frac{L}{\lambda} \right)^2 \left( \frac{1}{3} p_{in} + \frac{1}{6} p_{out} \right) \right] + \frac{1}{2} \left( \frac{L}{\lambda} \right)^2 \left[ p_{in} \left( 1 - \frac{1}{3} \frac{x}{L} \right) + \frac{1}{3} \frac{p_{out} x}{L} \right] \tag{3.34}
\]

From this equation it can be seen that the deviation from a linear pressure drop is controlled by the term \( \left( \frac{L}{\lambda} \right)^2 \). Using \( L = 0.3 \text{ m}, R = 6.5 \times 10^{-3} \text{ m}, w = 3.5 \times 10^{-3} \text{ m} \) and \( k = 1.7 \times 10^{-13} \text{ m}^2 \), the value of this parameter becomes

\[
\left( \frac{L}{\lambda} \right)^2 = 2.5 \times 10^{-4} \tag{3.35}
\]

This is obviously a small number, and thus the nonlinear terms in equation 3.34 can be neglected. It can then be concluded that the pressure drop through a channel with laminar flow can be assumed to be linear.

### 3.4 Porous material in channel and walls

It is now assumed that both the channel and the walls are porous materials in the flow problem shown in figure 3.2. The walls have the permeability \( k_w \) while the channel has the permeability \( k_c \) and \( k_c \gg k_w \). A section of the domain with the length \( dx \) is again considered, and it is assumed that Darcy’s law is valid in both the channel and the walls. The incoming flow rate, \( Q_p(x) \), is now found as

\[
Q_p(x) = -\frac{k_c \pi R^2}{\eta} p'(x) \tag{3.36}
\]

while the flow rate through the walls still is found from equation 3.28, but with \( k = k_w \). The mass conservation then becomes

\[
-\frac{k_c \pi R^2}{\eta} p'(x) = -\frac{k_c \pi R^2}{\eta} p'(x + dx) + \frac{2\pi R k_w}{\eta} \frac{\Delta p(x)}{w} dx \tag{3.37}
\]

The relation \( \frac{p'(x+dx)-p'(x)}{dx} \approx p''(x) \) is still valid and equation 3.37 is rewritten

\[
p''(x) = \frac{2k_w}{R w k_c} \Delta p(x) \tag{3.38}
\]
In the same way as before it is possible to rewrite the pressure equation

\[ p''(x) = \frac{1}{\lambda^2} \Delta p(x) \]  

(3.39)

where the constant \( \lambda \) this time is given as

\[ \lambda = \sqrt{\frac{R w k_c}{2 k_w}} \]  

(3.40)

The solution is given in the same way as in equation 3.32, 3.33 and 3.34, but with another \( \lambda \) value. Thus the term \( \left( \frac{L}{\lambda} \right)^2 \) again controls the deviation from linearity for the pressure drop. Using the parameters \( L = 0.3 \) m, \( R = 6.5 \times 10^{-3} \) m, \( w = 3.5 \times 10^{-3} \) m, \( k_w = 1.7 \times 10^{-13} \) m\(^2\) and \( k_c = 1 \) m\(^2\), the ratio \( \left( \frac{L}{\lambda} \right)^2 \) now becomes

\[ \left( \frac{L}{\lambda} \right)^2 = 1.5 \times 10^{-9} \]  

(3.41)

From this it is obvious that the pressure drop throughout the channel with good agreement is linear, when the ratio between the two permeabilities is very large.
Chapter 4

Simulations

This chapter presents the results from the simulations done in COMSOL. The chapter has been divided into two main parts: 2D and 3D simulations of the flow in the filter.

In the 2D simulations both a 7-channel filter and a 19-channel filter is studied. First the convergence of the solution is shortly examined on the 7-channel filter and hereafter the flow rate from the channels is found at different pressures. The filters are also modeled with porous material in both the substrate (the material between the channels) and the peripheral channels to see how this influences the flow through the filter.

In the 3D simulations the 7-channel filter and the 19-channel filter are studied again modeling the substrate and the channels as porous materials. The transmembrane flow rate is found at different sections down the length of the filter. Also the pressure in the inner channel and the outer channels is compared at different permeabilities.

4.1 2D simulations

As has been shown in section 3.2, the velocity in a porous material can be found using Darcy’s linear law, equation 3.24. In the simulations Darcy’s law is used on the following form

\[ \mathbf{v} = -K \nabla p \]  

(4.1)

where the constant \( K \) is called the coefficient of permeability, given as \( K = \frac{k}{\eta} \). This form of Darcy’s law is used with inspiration from [3].

The continuity equation for an incompressible fluid is given as

\[ \nabla \cdot \mathbf{v} = 0 \]  

(4.2)

By combining equation 4.1 and 4.2 the following is found

\[ \nabla \cdot (-K \nabla p) = 0 \quad \Rightarrow \quad \nabla^2 p = 0 \]  

(4.3)

which is the Laplace equation for the pressure. Thus the flow in the filter can be found by use of equations 4.3 and 4.1.
In the preliminary simulations the coefficient of permeability in the substrate, $K_s$, has been put to $K_s = 1 \text{ m}^2/(\text{Pa s})$ in order to simplify the calculations. This means that the velocity in the substrate is simply found as the gradient of the pressure, $v = -\nabla p$.

In the simulations with pressure in all the channels, a Dirichlet boundary conditions is used on both the rim of the filter and the boundaries of the channels. This means that the pressure is given at the boundaries in this situation.

It is shortly investigated if the solution for the 7-channel filter (shown in figure 2.1, pp. 3) converge with increasing number of elements, $N$, in the domain. This is done in order to check how accurate the solutions from the simulations are. During the convergence analysis the pressure in the inner channel, $p_i$, and the peripheral channels, $p_p$, is kept at $p_i = p_p = 1 \text{ Pa}$, while the pressure at the rim is given as $p_r = 0 \text{ Pa}$. The flow rate over the rim, $Q_r$, is used to estimate the error on the solution in the simulations. The flow rate is found by integrating the normal velocity over the boundary of the rim. The normal velocity, $v_n$, is found as

$$v_n = (\mathbf{v} \cdot \mathbf{n}) \mathbf{n}$$

where $\mathbf{n}$ is the normal vector to the boundary, pointing outwards. Since the present problem is a 2D problem, the boundary is only a line, not an area, giving a flow rate per meter. Thus the unit for the flow rate is $[\text{m}^3/(\text{m s})]$, shown like this in order to emphasize that it is a flow rate per meter.

![Figure 4.1: Convergence of the flow rate on the rim, $Q_r$, as function of number of elements, $N$.](image)

The error between two solutions is found as the absolute difference between the results divided by the result using the highest number of elements,

$$\varepsilon_{Q_r} = \frac{|Q_{r,2} - Q_{r,1}|}{Q_{r,2}}$$

(4.5)
4.1. 2D SIMULATIONS

where \( Q_{r,2} \) is found using a higher number of elements than \( Q_{r,1} \).

In figure 4.1 the error on the flow rate over the rim is plotted versus the number of elements, \( N \). The figure clearly shows that the error converges towards zero with increasing \( N \), as expected. Using around \( N = 1.8 \times 10^5 \) elements gives an error of \( \varepsilon_{Q_r} = 10^{-5} \). This is a good accuracy for these simulations, thus around \( 1.8 \times 10^5 \) elements are used in the rest of the 2D simulations.

4.1.1 7-channel filter

The flow in the filter is now investigated as the pressure in the peripheral channels is changed compared to the inner channel. Furthermore the flow rate over the boundary of the inner channel is compared to the flow rate over the boundary of the peripheral channels at the different pressures. Again the coefficient of permeability in the substrate is set to \( K_s = 1 \text{ m}^2/(\text{Pa s}) \) in the simulations.

In figure 4.2 the pressure distribution (figure 4.2(a)) and the velocity distribution (figure 4.2(b)) is shown for \( p_i = p_p = 1 \text{ Pa} \) in the channels and \( p_r = 0 \text{ Pa} \) on the rim. The two figures clearly show, that the peripheral channels create a ”screening pressure” around the inner channel such that the pressure between the peripheral channels and the inner channel is almost constant. As it is the pressure difference, that drives the flow through the filter, the consequence is that the flow from the inner channel is almost blocked completely, when the pressure is the same in the channels. This can also be seen on figure 4.2(b) where it is nearly impossible to see the arrows of the normal velocity on the boundary of the inner channel. Furthermore the velocity between the inner channel and the peripheral channels is very small. The flow rate over the boundary of the inner channel is \( Q_i = 0.14 \text{ m}^3/(\text{m s}) \), while it is \( Q_p = 1.65 \text{ m}^3/(\text{m s}) \) over the boundary of one of the peripheral channels. The flow rate from the inner channel is then less than 10% of the flow rate from one of the peripheral channels.

![Pressure distribution](image1.png) ![Velocity distribution](image2.png)

(a) Pressure distribution with arrow plot of the velocity.
(b) Velocity distribution with arrow plot of the normal velocity on the boundaries.

Figure 4.2: Pressure and velocity distribution in the filter with \( p_i = p_p = 1 \text{ Pa} \) in the channels and \( p_r = 0 \text{ Pa} \) on the rim.
It can also be seen from figure 4.2(b) that the velocity in the flow is significantly higher on the outside of the peripheral channels, than anywhere else in the filter. This makes sense, since this is where there must be the biggest pressure gradient. On the backside of the peripheral channels on the other hand, the flow over the boundary is very small, which is due to the pressure in the inner channel.

In figure 4.3 the pressure in the peripheral channels has been changed to \( p_p = 0.8 \) Pa, while the pressure in the inner channel and the pressure on the rim is kept at \( p_i = 1 \) Pa and \( p_r = 0 \) Pa, respectively. As the pressure in the peripheral channels has been lowered, the screening of the inner channel has decreased compared to before. The two figures in figure 4.3 clearly show, though, that the peripheral channels still block the inner channel to some extend. The flow over the boundary of the inner channel has increased, which can be seen from figure 4.3(b) where the arrows of the normal velocity on the boundary of the inner channel are now clearly seen. The velocity between the peripheral channels and the inner channel is still very low, though, which is due to the screening. The flow rate from the inner channel is \( Q_i = 0.95 \) m\(^3\)/s, while it is \( Q_p = 1.19 \) m\(^3\)/s from one of the peripheral channels. Lowering the pressure in the peripheral channels from 1 Pa to 0.8 Pa has then improved the effect of the inner channel to 80% compared to the peripheral channels, which is a significant improvement. This improvement is not only due to an increase in the flow rate from the inner channel but also a decrease in the flow rate from the peripheral channels, creating a double effect when comparing the two.

![Figure 4.3: Pressure and velocity distribution in the filter with \( p_i = 1 \) Pa in the inner channel and \( p_p = 0.8 \) Pa in the peripheral channels.](image)

The flow from the peripheral channels does not seem to have changed much by lowering the pressure, when figure 4.3 is compared with figure 4.2. On the outside of the channels the velocity is still the highest in the filter, though it has decreased a bit. On the backside of the channels the velocity is still very small, but now the flow is going into the channels, instead of going out of the channels. This change is due to the pressure difference between the inner channel and the peripheral channels, creating a flow from the inner channel into...
the peripheral channels thus making the flow rate from the peripheral channels smaller.

In figure 4.4 the pressure in the peripheral channels has been lowered to \( p_p = 0.6 \text{ Pa} \). From figure 4.4(a) it is now clear that the inner channel is the dominating channel in the filter. The peripheral channels do not create a screening pressure anymore. Figure 4.4(b) shows that the normal velocity on the boundary of the inner channel is now almost as high as the velocity on the outside of the peripheral channels. Furthermore a significant amount of flow is now going from the inner channel and into the peripheral channels. The flow rate from the inner channel is now \( Q_i = 1.76 \text{ m}^3/(\text{m s}) \) and from one of the peripheral channels it is \( Q_p = 0.72 \text{ m}^3/(\text{m s}) \). This means that the flow rate from the inner channel is now almost 2.5 times bigger than the flow rate from a peripheral channel. This difference is both due to the fact that the pressure is now much higher in the inner channel than in the peripheral channels, and that there is now a significant amount of flow into the peripheral channels, lowering the combined flow rate from these channels and making them less efficient.

\[ (a) \text{ Pressure distribution with arrow plot of the velocity.} \]
\[ (b) \text{ Velocity distribution with arrow plot of the normal velocity on the boundaries.} \]

Figure 4.4: Pressure and velocity distribution in the filter with \( p_i = 1 \text{ Pa} \) in the inner channel and \( p_p = 0.6 \text{ Pa} \) in the peripheral channels.

In table 4.1 the flow rates from the channels and the rim has been listed for different pressures in the peripheral channels, while the pressure in the inner channel and the pressure on the rim is kept at \( p_i = 1 \text{ Pa} \) and \( p_r = 0 \text{ Pa} \), respectively. The table shows that the flow rates changes linearly with the pressure in the peripheral channels. In the last two columns the flow rate from the inner channel is compared to the flow rate from a peripheral channel and the flow rate over the rim. From these columns it can be seen, that when the pressure in the peripheral channels is lowered to \( p_p = 0.5 \text{ Pa} \), which is half the pressure in the inner channel, the flow rate from the inner channel is 4.5 times higher than the flow rate from a peripheral channel, thus the inner channel will deliver more than 40% of the flow in the filter. That means that the peripheral channels are very inefficient at this low a pressure, which partly is due to a high flow from the inner channel into the peripheral channels, see figure 4.4(b).
On the other hand when the pressure in the peripheral channels equal the pressure in the inner channel, the inner channel becomes highly inefficient only delivering 1.4% of the total flow rate.

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
p_p [\text{Pa}] & Q_i [\text{m}^3/(\text{m s})] & Q_p [\text{m}^3/(\text{m s})] & Q_r [\text{m}^3/(\text{m s})] & Q_i/Q_p [%] & Q_i/Q_r [%] \\
\hline
1 & 0.14 & 1.65 & 10.05 & 8.6 & 1.4 \\
0.9 & 0.55 & 1.42 & 9.06 & 38.5 & 6.0 \\
0.8 & 0.95 & 1.19 & 8.07 & 80.2 & 11.8 \\
0.7 & 1.36 & 0.95 & 7.08 & 142.2 & 19.2 \\
0.6 & 1.76 & 0.72 & 6.09 & 244.1 & 28.9 \\
0.5 & 2.17 & 0.49 & 5.10 & 443.1 & 42.5 \\
\hline
\end{array}
\]

Table 4.1: The flow rates from the channels and the rim at different pressures in the peripheral channels while the pressure in the inner channel and on the rim is kept constant at \(p_i = 1 \text{ Pa}\) and \(p_r = 0 \text{ Pa}\), respectively.

4.1.2 7-channel filter with porous channels

As seen in section 3.4 it should be possible to model the channels as porous material, just like the substrate, which would make it easier to model the problem in COMSOL. In order to see if it is a valid way to model the domain, the peripheral channels are now made porous. This means that the boundary condition on these channels has to be changed. Before a Dirichlet boundary condition was used, which meant that the pressure was determined on the boundary. This is no longer possible as these boundaries have become internal boundaries in the domain. Thus it is necessary to use a Neumann boundary condition, determining that the normal velocity across the boundary must be continuous

\[
n \cdot v_p = n \cdot v_s
\]  

(4.6)

where \(n\) is the normal vector the boundary and \(v_p\) and \(v_s\) are the velocities in the peripheral channels and the substrate, respectively. The two velocities are found using Darcy’s law, equation 4.1, whereby the velocities can be expressed as

\[
\begin{align*}
v_p &= -K_c \nabla p \\
v_s &= -K_s \nabla p
\end{align*}
\]

(4.7)

where \(K_c\) and \(K_s\) are the coefficient of permeability for the channels and the substrate, respectively. \(K_c\) is kept at \(K_c = 1 \text{ m}^2/(\text{Pa s})\) through all the 2D simulations. The pressure in the inner channel and on the rim is still \(p_i = 1 \text{ Pa}\) and \(p_r = 0 \text{ Pa}\).

First the two coefficients of permeability are set to \(K_c = K_s = 1 \text{ m}^2/(\text{Pa s})\) which means that the channels should have no influence on the flow in the filter, since the channels and the substrate have the same permeability. This situation has been plotted in figure 4.5. As the figure clearly shows, the peripheral channels do not influence the flow. This can both be seen from the uniformly distributed pressure and the linear streamlines of the velocity going from the inner channel to the rim.
4.1. 2D SIMULATIONS

Figure 4.5: Pressure distribution in the filter with streamlines of the velocity. Pressure in the inner channel is $p_i = 1 \text{ Pa}$ and $p_r = 0 \text{ Pa}$ on the rim. The constant of permeability in the peripheral channels and in the substrate is $K_c = K_s = 1 \text{ m}^2/(\text{Pa s})$.

In figure 4.6 the channels are made more permeable than the substrate with a ratio of $\frac{K_c}{K_s} = 10^3$. The pressure distribution is plotted in figure 4.6(a) and the velocity distribution is plotted in figure 4.6(b). From these figures it is obvious that the flow is trying to go through the channels on its way from the inner channel to the rim. This seems sensible, since it must be easier for the flow to go through the channels, than through the substrate, when the channels have a higher permeability than the substrate.

Figure 4.6: Pressure and velocity distribution in the filter with $\frac{K_c}{K_s} = 10^3$. The peripheral channels then have a higher permeability than the substrate.

From figure 4.6(b) it can also be seen, that the flow has a higher velocity in the channels than in the substrate due to the higher permeability and thereby lower resistance in the channels. Making the $\frac{K_c}{K_s}$-ratio higher does not change the flow behavior, and plots of the
flow at higher ratios have thus not been included in the report.

In figure 4.7 the situation is then reversed and the channels are now less permeable than the substrate with a ratio of $\frac{K_c}{K_s} = 10^{-1}$. This should make the channels function as walls in the flow due to higher resistance in the channels. This would of course not be the case in a typical filter, but the situation is nevertheless studied, in order to see if the flow behaves as expected.

The figure shows, that the channels do indeed block the flow now. In figure 4.7(a) the streamlines show that the flow is going around the channels, and in figure 4.7(b) it can be seen that the velocity in the channels is almost zero, while it is higher on the sides of the channels, indicating that the flow is accelerated around the channels. Again, changing the $\frac{K_c}{K_s}$-ratio does not change the flow behavior and thus plots of the flow at other ratios have been excluded from the report.

4.1.3 19-channel filter

The flow in the 19-channel filter is now investigated as the pressure in the peripheral and intermediate channels is changes compared to the inner channel. The peripheral channels in the 7-channel filter are moved from $R_2 = 0.5$ m to $R_2 = 1/3$ m and a new ring of channels is placed in a distance of $R_3 = 2/3$ m from the center. This new peripheral ring contains 12 channels placed with $\theta = 30^\circ$ between them, i.e channels are placed at the same angles as the intermediate placed channels (peripheral channel 1) and one in between these (peripheral channel 2), see figure 4.8. The coefficient of permeability in the substrate is set to $K_s = 1$ m$^2$/(Pa s) in these simulations.

In figure 4.9 the pressure distribution (figure 4.9(a)) and the velocity distribution (figure 4.9(b)) is shown for $p_i = p_{int} = p_p = 1$ Pa in the channels and $p_r = 0$ Pa on the rim. $p_{int}$ is the pressure in the intermediate channels. The two figures clearly show, that the peripheral ring of channels create a "screening pressure" around the intermediate and
4.1. 2D SIMULATIONS

Figure 4.8: The geometry of a symmetry section of the 19-channel filter. The names of the different placed channels are also shown.

![Image](image_url)

(a) Pressure distribution with arrow plot of the velocity.
(b) Velocity distribution with arrow plot of the normal velocity on the boundary.

Figure 4.9: Pressure and velocity distribution in the filter with $p_i = p_{int} = p_p = 1$ Pa in the channels and $P_r = 0$ Pa on the rim.

inner channels in the same way as the peripheral ring of channels did in the 7-channel filter. The pressure is again almost constant from the inner channel to the peripheral channels, i.e. the velocity is nearly zero as figure 4.9(b) also shows.

The flow rate over the boundary of the inner channel is $Q_i = 3.33 \cdot 10^{-3}$ m$^3$/m s, the flux over the boundary of one intermediate channel is $Q_{int} = 1.02 \cdot 10^{-2}$ m$^3$/m s and the flux for one peripheral channel is $Q_{p1} = 1.68$ m$^3$/m s. The flux in peripheral channel 1 and 2 is practically the same in this situation. The flow rate from the inner channel is less then 0.02 % of the flow rate from one of the peripheral channels, while the flow rate from one of the intermediate channels is around 6 % of the flow rate from one of the peripheral channels. The flux from the inner channel is thus insignificant when $p_i = p_{int} = p_p = 1$, and could be omitted without any loss and also the intermediate channels have nearly no contribution at all.

In figure 4.10 the pressure in the intermediate and peripheral channels have been changed to $p_{int} = 0.9$ Pa and $p_p = 0.8$ Pa, respectively. The pressure in the inner channel and on the rim is unchanged. These two figures still show that the peripheral channels block the inner and intermediate channels to some extend. The flow over the boundary
of the inner and intermediate channels have increased, which can be seen from the normal velocity arrows in figure 4.10(b). The flow rate from the inner channel is $Q_i = 0.67 \text{ m}^3/(\text{m s})$, the flux over the boundary of an intermediate channel is $Q_{int} = 0.16 \text{ m}^3/(\text{m s})$ and $Q_{p1} = 6.19 \text{ m}^3/(\text{m s})$ from one of the peripheral channels. Lowering the pressure in the intermediate and peripheral channels has improved the effect of the inner channel to 56 % compared to the peripheral channels and improved the effect of the intermediate channel to 13 % compared to the peripheral channels. This is quite a dramatic improvement of the inner channel, which had no contribution at all before, while the improvement of the intermediate channels is not nearly as big. This is because fluid from the inner channel flows into the intermediate channel thus making it less efficient. The same happens from the intermediate to the peripheral channels. This was also seen for the 7-channel filter.

In figure 4.11 the pressure in the intermediate channels is lowered to $p_{int} = 0.8 \text{ Pa}$ and the pressure in the peripheral channels to $p_p = 0.6 \text{ Pa}$. The pressure in the inner channel and on the rim is unchanged. As the figure shows, the peripheral channels do not create a screening pressure anymore. A significant amount of flow is now going from the inner channel into the intermediate channels, as well as from the intermediate channels into the peripheral channels. The flow rate from the inner channel is now $Q_i = 1.34 \text{ m}^3/(\text{m s})$, the flow rate from an intermediate channel $Q_{int} = 0.30 \text{ m}^3/(\text{m s})$ and $Q_{p1} = 0.72 \text{ m}^3/(\text{m s})$ from one of the peripheral channels. The flow rate from the inner channel is now 2 times greater than the flow rate from a peripheral channel and the flow rate from one intermediate channel is 42 % of the flow rate from a peripheral channel.

In table 4.2 the flow rate leaving the channels is compared with the total transmembrane flux, for different pressures in the intermediate and peripheral channels. The flow rate from the peripheral channels are divided into two, the channels located behind the intermediate channels, $Q_{p1}$ and channels located between the intermediate channels, $Q_{p2}$, see figure 4.8. The pressure in the inner channel and on the rim is kept constant at $p_i = 1 \text{ Pa}$ and $p_r = 0 \text{ Pa}$. The table shows that the flow rate from the inner and intermediate
4.2 3D simulation

This section presents the result from the 3D simulations done in COMSOL. The section is divided into two subsection describing the results from simulations for the 7-channel filter and the 19-channel filter. In all the 3D simulations, both the channels and the substrate are modeled as porous materials.

(a) Pressure distribution with arrow plot of the velocity. (b) Velocity distribution with arrow plot of the normal velocity on the boundary.

Figure 4.11: Pressure and velocity distribution in the filter with $p_i = 1$ Pa, $p_{int} = 0.8$ Pa, $p_p = 0.6$ Pa in the channels and $p_r = 0$ Pa on the rim.

<table>
<thead>
<tr>
<th>$p_p$ [Pa]</th>
<th>$p_{int}$ [Pa]</th>
<th>$Q_i/Q_r$ [%]</th>
<th>$Q_{int}/Q_r$ [%]</th>
<th>$Q_p1/Q_r$ [%]</th>
<th>$Q_p2/Q_r$ [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1.64 $\cdot$ 10$^{-4}$</td>
<td>5.04 $\cdot$ 10$^{-2}$</td>
<td>8.29</td>
<td>8.31</td>
</tr>
<tr>
<td>0.8</td>
<td>0.9</td>
<td>4.16</td>
<td>0.97</td>
<td>7.41</td>
<td>7.57</td>
</tr>
<tr>
<td>0.6</td>
<td>0.8</td>
<td>11.09</td>
<td>2.50</td>
<td>5.96</td>
<td>6.34</td>
</tr>
</tbody>
</table>

Table 4.2: The flow rate from the channels compared to the flow total flow rate, at different pressure in the peripheral and intermediate channels, while the pressure in the inner channel and on the rim is kept constant at $p_i = 1$ Pa and $p_r = 0$ Pa, respectively.

channels are insignificant when the pressure in the channels are equal. This was also seen in the 7-channel filter, but for the 19-channel filter the effect is even greater. Further the table shows that the flow rate from the inner channel is twice the flow rate from one peripheral channel when the pressure in the intermediate and the peripheral channels are lowered.

From this it can then be seen, that when the pressures in the channels are equal, only the peripheral channels contribute to the flux. A pressure difference between the channels has to be created, if the channels behind the peripheral channels shall have any effect.
4.2.1 7-channel filter

The filter with 7 channels has a geometry as shown in figure 2.1 and figure 2.2. It has the same dimensions as the 2D 7-channel filter and a length of $L = 5$ m. This model is both modeled in a full version and a symmetry version. In the symmetry version only one twelfth of the filter is modeled. In this way a finer mesh can be used, i.e. a more accurate solution is achieved. In the following the plots are extracted from the full model and the flux values from the symmetric version.

At the inlet to the channels a Dirichlet boundary condition is used whereby the pressure is kept constant at $p_{in} = 1$ Pa. On the inlet surface to the substrate, however, a Neumann boundary condition is used. Hereby the velocity at the surface of the substrate is kept equal to zero, i.e. the substrate surface is functioning as a wall. This means that the flow is forced into the channels and cannot go into the substrate at the inlet of the filter. At the end of the filter, the outlet, a Dirichlet boundary condition is used on both the substrate and the channels. Also at the surface of the substrate (the outside of the filter) a Dirichlet boundary condition is used. The pressure at the outlet and at the surface of the substrate is kept at $p_{out} = p_s = 0$ Pa.

At the internal boundaries between the channels and the substrate a Neumann boundary condition is used to ensure that the flow velocity across the boundaries is continuous, i.e. $n \cdot v_c = n \cdot v_s$, where $n$ is the normal vector, $v_s$ is the velocity in the substrate and $v_c$ is the velocity in the channels.

The coefficient of permeability in the substrate is called $K_s$ and $K_c$ in the channels. $K_c$ is fixed at $K_c = 1 \text{ m}^2/(\text{Pa s})$ through all the simulations. The coefficient of permeability in the substrate is varied in the interval $K_s = [1; 10^{-4}] \text{ m}^2/(\text{Pa s})$.

The pressure is plotted versus the length of the filter in figure 4.12. The solid lines indicates that the pressure is evaluated in the center of the filter, i.e. in the center of the inner channel and the dashed line is the pressure in the center of one of the peripheral

![Figure 4.12](image_url)

**Figure 4.12:** Pressure distribution along the length of the filter. The solid lines is the pressure in the center of the inner channel while the dashed lines indicate that the pressure is evaluated in the center of a peripheral channel. 1: $\frac{K_c}{K_s} = 1$, 2: $\frac{K_c}{K_s} = 10^1$, 3: $\frac{K_c}{K_s} = 10^2$, 4: $\frac{K_c}{K_s} = 10^3$ and 5: $\frac{K_c}{K_s} = 10^4$
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The figure shows that the pressure drops exponentially when the permeabilities in the substrate and the channels are equal (the blue lines). Figure 4.12 also shows that the pressure drop through the filter becomes more and more linear as the $\frac{K_c}{K_s}$-ratio is increased. When the ratio between the permeability coefficients is $10^4$, the pressure drop is very close to be linear through the filter. The figure shows further that a pressure difference between

\[
\begin{align*}
(a) & \quad \frac{K_c}{K_s} = 1. \\
(b) & \quad \frac{K_c}{K_s} = 10^1. \\
(c) & \quad \frac{K_c}{K_s} = 10^2. \\
(d) & \quad \frac{K_c}{K_s} = 10^3. \\
(e) & \quad \frac{K_c}{K_s} = 10^4. \\
(f) & \quad \frac{K_c}{K_s} = 10^5. 
\end{align*}
\]

Figure 4.13: The pressure distribution in a cross-section of the filter at different $\frac{K_c}{K_s}$-ratios.

the inner channel and the peripheral channels is present. As the $\frac{K_c}{K_s}$-ratio increases, the pressure difference increases until $\frac{K_c}{K_s} = 10^3$ where the pressure difference attains its maximum. Hereafter the difference decreases again with growing ratio, as both the pressure drops approach a linear distribution through the filter. As seen in the 2D simulations the pressure difference between the inner channel and the peripheral channels will ensure that the inner channel contributes more to the transmembrane flow rate. In the two ends of the filter, the pressure in the inner channels and the peripheral channels are equal due to the boundary conditions.
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(a) Flux through the three boundaries.

Figure 4.14: Flux through the three boundaries with $\frac{K_c}{K_s} = 1$.

(b) Zoom of figure 4.14(a).

The pressure distribution is plotted in a cross section of the filter for different $\frac{K_c}{K_s}$-ratios in figure 4.13. The same tendencies for the pressure as seen in figure 4.12, is seen in figure 4.13. The pressure is exponential decreasing for low permeability ratios, while it becomes more and more linearly distributed as the ratio is increased. In figures 4.13(e) and 4.13(f) the pressure drop is linear in the center of the filter. In figures 4.13(a)-4.13(d) the pressure is higher in the inner channel compared to the peripheral channels, just as seen in figure 4.12.

The total transmembrane flux through the surface of the substrate, the flux out of the inner channel and the flux out of a peripheral channel is found. The flux is estimated in the same way as done in the 2D simulations; by integrating the normal velocity component at the boundary. These three flow rates are evaluated in five sections of the filter. The first section contains the first fifth of the length, the next section is the second fifth, and so forth. In figure 4.14 the flux through the three boundaries is plotted in the five sections.

(a) Flux through the three boundaries.

Figure 4.15: Flux through the three boundaries with $\frac{K_c}{K_s} = 10^1$.

(b) Zoom of figure 4.15(a).
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along the length of the filter for \( \frac{K_c}{K_s} = 1 \). The figure shows that nearly all the flux through the three boundaries is located in the two first sections. The reason is that \( \frac{K_c}{K_s} = 1 \), which means that the material in the channels and substrate has the same permeability. Thus there is no difference between the channels and the substrate. Since it is a shorter way from the channels to the surface of the substrate, than to the end of the filter, the flow will quickly dissipate out of the channels in the start of the filter.

In figure 4.14(b), which is a zoom of figure 4.14(a), it is clear that the flux through the inner channel and the peripheral channels is nearly zero in the three last sections of the filter. The flux leaving the inner channel is larger than the amount leaving the peripheral channels, which is caused by the pressure difference between these two channels, see figure 4.12.

The flux through the three boundaries is plotted in figure 4.15 for \( \frac{K_c}{K_s} = 10^1 \). The

Figure 4.16: Flux through the three boundaries with \( \frac{K_c}{K_s} = 10^2 \).

Figure 4.17: Flux through the three boundaries with \( \frac{K_c}{K_s} = 10^3 \) and the flux leaving the channels at the outlet, at different permeability ratios.
flux leaving the filter in the first section is about 5 times smaller than for $\frac{K_c}{K_s} = 1$. This is caused by the lower permeability in the substrate, i.e. the resistance is higher in the substrate than in the channels. With this permeability ratio, section three also has a contribution to the transmembrane flux on the surface of the substrate, but it is still in the first section that the largest flux is present. Both in the second and third section a larger flux is leaving the inner channel compared to a peripheral channel, see figure 4.15(b). This is again caused by the pressure difference between these two channels, see figure 4.12.

Figure 4.16 shows the flux through the three boundaries for $\frac{K_c}{K_s} = 10^2$. The total flux leaving the filter is again smaller, caused by the lower permeability in the substrate. At this ratio nearly a linearly decreasing flux is observed through the sections, the reason is a more linear pressure drop through the filter, see figure 4.12. In the three last sections, the flux leaving the inner and a peripheral channel are very close to each other, see the

Figure 4.18: The absolute velocity in a cross section of the filter at different $\frac{K_c}{K_s}$-ratios.
4.2. 3D SIMULATION

The flux over the three boundaries is plotted in figure 4.16(a) for $\frac{K_c}{K_s} = 10^3$. The magnitude of the total transmembrane flux out of the surface of the substrate has again decreased compared to the lower permeability ratios and it is now close to zero even in the start of the filter. This is because the pressure difference across the filter is not large enough to overcome the resistance in the substrate compared to the resistance in the channels. The fluid stays inside the channels and leave the filter at the channels outlet in the end of the filter. In figure 4.17(b) the flux leaving the channels at the filter outlet is plotted versus the $\frac{K_c}{K_s}$-ratio. This figure clearly shows that the flux leaving the channels at the outlet of the filter increases fast as the ratio of permeability is increased.

In figure 4.18 the absolute velocity in a cross section of the filter is plotted for the different permeability ratios. The figure shows that the flow in the channels become more dominant with higher permeability ratio, just as it is expected because the higher the $\frac{K_c}{K_s}$-ratio, the harder it is for the flow to leave the channels through the walls. At the highest permeability ratios, the velocity becomes constant in the channels, caused by the constant pressure gradient shown in figure 4.12. Further it can be seen that for the low $\frac{K_c}{K_s}$-ratios the absolute velocity in the peripheral channels is higher than in the inner channels at the inlet due to a higher velocity from the channels into the substrate. In the middle of the filter, on the other hand, the magnitude of the velocity in the inner channel is a bit greater than in the peripheral channels, just as it was seen for the flux out of the channels.

4.2.2 19-channel filter

A 3D simulation of a filter with 19 channels is made in COMSOL, in the same way as the 3D simulations of the 7-channel filter where made. The geometry is the same as the 2D filter with 19 channels and has a length of $L = 5$ m. Only a twelfth of the filter is modeled due to symmetry. The same boundary conditions as was used for the 7-channel filter in 3D are used here and the coefficients of permeability are also varied in the same way.

The pressure in a cross section is plotted in figure 4.19 for different $\frac{K_c}{K_s}$-ratios. The same tendencies are seen in figure 4.19, as in figure 4.13, pp. 25, for the 7-channel filter. The pressure is exponentially decreasing through the length of the filter at low permeability ratios, but becomes more and more linear for higher ratios. A pressure difference from the peripheral channels to the inner channel and intermediate placed channels are visible for low ratios. This pressure difference is larger than the pressure difference seen for the 7-channel filter, between the inner and peripheral channel. A pressure difference is also present between the intermediate channels and the inner channel, but this is very small compared to the pressure difference from the peripheral channels to the intermediate channels.

The flux through the boundaries are found in five sections down the length of the filter in the same way as for the 7-channel filter. Now five fluxes are calculated due to the two new channels, peripheral channel 1 and peripheral channel 2, see figure 4.8.

In figure 4.20 the flux through the five boundaries in the five sections is plotted for $\frac{K_c}{K_s} = 1$. The flux is still dominant in the first section, as it was for the 7-channel filter. The amount of flux leaving the peripheral channels, both 1 and 2, is greater than the
flux leaving the intermediate and the inner channel. The two peripheral channels have nearly the same flux, while the flux from the intermediate channels is greater than the flux leaving the inner channel. In the second and third section the largest amount of flux is flowing through the inner channel, while the two peripheral channels have the lowest amount. This is caused by the pressure difference between these channels, as seen in figure 4.19.

The flux through the five boundaries for \( \frac{K_c}{K_s} = 10^1 \) is plotted in figure 4.21. Here the

![Figure 4.19: The pressure distribution in a cross section of the 19-channel filter at different \( \frac{K_c}{K_s} \)-ratios.](image)

![Figure 4.20: Flux through the five boundaries with \( \frac{K_c}{K_s} = 1 \).](image)
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same tendencies are seen, but the amount of flux leaving the filter is 5 times smaller than for $\frac{K_c}{K_s} = 1$ due to the higher resistance in the substrate. As expected the third section now has a higher flow rate than before, just as it was seen for the 7-channel filter. From the zoom in figure 4.21(b) it can be seen that the flux from the two peripheral channels is again close to each other. And again the flux from the inner channel and the intermediate channels is higher than the flux from the peripheral channels in section two and three due to the pressure difference between the channels.

In figure 4.22 the flux through the five boundaries is plotted for $\frac{K_c}{K_s} = 10^2$. The flux is now more linearly distributed than for the lower permeability ratios. The amount of flux is a factor 6 smaller than at a permeability ratio of $10^1$, due to the lower permeability in the substrate. The zoom in figure 4.22(b) shows that the flux from the inner channel and the intermediate channels actually increases from the first to the second section, though they still contribute less than the peripheral channels in the second section. After the second section, all the channels has more or less the same flux.

Figure 4.21: Flux through the five boundaries with $\frac{K_c}{K_s} = 10^1$.

Figure 4.22: Flux through the five boundaries with $\frac{K_c}{K_s} = 10^2$. 
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Figure 4.23(a) shows the flux for $K_c/K_s = 10^3$. Now the total flux is almost linearly decreasing through the five sections as it did in the 7-channel filter. The flow leaving the substrate is close to zero due to the high resistance in the substrate compared to the pressure drop across the substrate. The fluid stays in the channels and leave the filter at the outlet of the channels, as seen in figure 4.23(b). Here the flux leaving the channels is plotted for the different permeability ratios and the figure clearly shows that the flux leaving the channels at the filter outlet is rising as the ratio grows.

(a) Flux through the three boundaries with $K_c/K_s = 10^3$.

(b) Flux leaving the channels at the outlet of the filter.

Figure 4.23: Flux through the five boundaries with $K_c/K_s = 10^3$ and the flux leaving the channels at the outlet at the different permeability ratios.

The total transmembrane flux leaving the filter from the substrate, is plotted in figure 4.24 versus the permeability ratios, both for the 7-channel filter and the 19-channel filter. From the figure it can be seen that the flux leaving the 19-channel filter is much greater than the flux from the 7-channel filter at low permeability ratios. Increasing the number of channels in the filter, thus makes the filter more efficient at low ratios. When the permeability ratio is high, however, the two filters have more or less the same transmembrane flux. This is because the resistance in the substrate becomes so high, that almost no flow leaves the channels. Thus it does not matter how many channels, the filter contains. From this it can be seen that the extra channels only have an effect on the transmembrane flux at low permeability ratios.
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Figure 4.24: The total transmembrane flux leaving the 7-channel filter and the 19-channel filter for different $\frac{k_c}{k_s}$-ratios.
Chapter 5

Conclusion

The theoretical analysis showed, that the flow in a porous material can be described by Darcy’s linear law for the velocity. Furthermore it was showed that it should be possible to model both channels and substrate in a filter as porous materials, as long as the permeability in the channels is very large compared to the permeability in the substrate.

From the 2D simulations it could be seen that when the pressure in the inner and peripheral channels were equal, the pressure from the peripheral channels functioned as a ”screening pressure” on the inner channel thus making the inner channel highly inefficient in the 7-channel filter only delivering 1.4% of the total flux. This effect was seen to an even greater extend for the 19-channel filter where both the inner channel and the intermediate channels where screened by the peripheral channels. Here both the inner channel and the intermediate channels contributed with nothing at all and practically all the flux came from the peripheral channels. It can then be concluded that when the pressure in the channels are equal, almost all the transmembrane flux is delivered from the peripheral channels, and the efficiency of the filter is therefore not increased significantly by inserting another ring of channels. Also this points towards that the efficiency of the filter only depends on the number of channels in the peripheral ring.

When the pressure in the peripheral channels was lowered, the flux from the inner channel rose as the screening decreased. It could also be seen, that if the pressure in the peripheral channels got low enough, the flow from the inner channel started to go into the peripheral channel, thus making them less efficient. Again this effect was seen for both filters.

The 3D simulation showed a constant pressure gradient in the channels in the length of the filter for high \(\frac{K_c}{K_s}\)-ratios, while the pressure gradient decreased exponentially in the channels for low \(\frac{K_c}{K_s}\)-ratios. This was because when the permeability ratio was low, the simulations showed that the flow in the present case quickly dissipated out of the channels into the substrate in the start of the filter, but when the ratio was increased, the resistance in the substrate become so high, that almost no flow left the channels.

The transmembrane flow rate leaving the channels was plotted in five sections along the length of the filter. The plots showed that for low \(\frac{K_c}{K_s}\)-ratios all the flux out of the
filter was situated in the first and second section. As the ratios was increased, the total transmembrane flux approached a linear distribution along the length. This effect is again due to the increasing resistance in the substrate as described above. Looking at the flux from the different channels showed that for the low permeability ratios, the inner and intermediate channels had a higher flux than the peripheral channels in the second and third section due to a pressure difference between the peripheral and inner channels. As the ratio was increased, this pressure difference disappeared and therefore nearly all the flux came from the peripheral channels.

The total flux from the 7-channel filter and the 19-channel filter was compared at the different permeability ratios. This showed that when the ratio was low, the 19-channel filter had a significantly higher efficiency than the 7-channel filter. As the ratio was increased, the difference became smaller until they were almost equal, as they both dropped towards zero flux out of the filter. From this it can be concluded that the present simulations show that the efficiency of the filter more or less does not increase with the number of channels for high permeability ratios.
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